TOWARDS GLOBAL CLASSIFICATIONS: A DIOPHANTINE APPROACH

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An open problem is the global classification, up to linear transformations, of polynomial two-component integrable equations $(u_t, v_t) = K$,

$$\begin{aligned} u_t &= au_n + K_1^{i_1,1-i_1} + \cdots + K_1^{i_r,1-i_r} + \cdots \\ v_t &= bv_n + K_2^{1-i_{r+1},i_{r+1}} + \cdots + K_2^{1-i_s,i_s} + \cdots . \end{aligned} \tag{1}$$

where $i_j \in \{-1, 0, 1\}$, the $K_k^{i,j}$ are nonzero polynomials, of degree $i+2-k$ in the variables $(u_0{=}u, u_1{=}u_x, \ldots)$ and of degree $j+k-1$ in $(v_0{=}v, v_1{=}v_x, \ldots)$, and the dots refer to terms of higher degree. The word global means we aim for a complete description at any order $n \in \mathbb{N}$. A pair of functions S is called an *approximate symmetry* of (1) if the Lie-bracket $[K, S]$ vanishes modulo cubic or higher degree terms.[1] And the existence of infinitely many approximate symmetries is taken as the definition of approximate integrability. We contribute to the above mentioned problem by globally classifying approximately integrable equations (1). Note that an equation may have infinitely many approximate symmetries, but fail to have any symmetries. That problem involves conditions of higher degree and is left open.

In the symbolic method, the $K_k^{i,1-i}$ are transformed into polynomials in just two symbols. The action of the Lie-derivative $[K^{0,0},]$ on homogeneous quadratic parts of S is represented by two (related) sets of polynomials (k=1,2) in the two symbols, of total degree n: $[K^{0,0}, S^{i,j}]_k{=}\mathcal{G}_{k;n}^{i,j}[a{:}b]S_k^{i,j}$.[2] We denote the tuple $K_1^{i_1,1-i_1}, \ldots, K_2^{1-i_s,i_s}$ by K^1. Suppose $a{\neq}b$. Then an approximate symmetry exists iff the tuple $S^1 = \mathcal{G}_m[c{:}d]K^1/\mathcal{G}_n[a{:}b]$ consist of polynomials (with the right symmetry properties).

A *necessary* condition for equation (1) to be approximately integrable, is the existence of a proper tuple H dividing infinitely many $\mathcal{G}_m[c{:}d]$, including $m{=}n$ for $c/d{=}a/b$. Using the Skolem–Mahler–Lech theorem, results on diophantine equations involving roots of unity obtained by Beukers,[3] and an algorithm of C.J. Smyth, we have classified the divisors of infinitely many $\mathcal{G}$-tuples, with $s = 1, 2$, together with the orders $m \in \mathbb{N}$ and values $c/d \in \mathbb{C}$ at which they appear.[4] Using these results one can write down the set of divisors of infinitely many $\mathcal{G}$-tuples with $2 < s < 7$. The main point we stress here is that this knowledge is also *sufficient*.

Denote by $m(H)$ the set of orders m such that there exist $c, d \in \mathbb{C}$ for which $H \mid \mathcal{G}_m[c{:}d]$ and no $\mathcal{G}$-function vanishes. And denote by $\mathcal{H}_n$ the set of proper tuples H with infinite $m(H)$, whose smallest element is n.

Lemma: *Let $H \in \mathcal{H}_n$ and $a, b \in \mathbb{C}$ such that $H \mid \mathcal{G}_n[a{:}b]$. Let $P \in \mathcal{H}_n$ be the tuple of highest degree such that $H \mid P$ and $m(H) = m(P)$. Then equation (1) is approximately integrable, with symmetries at orders $m \in m(H)$, if $\mathcal{G}_n[a{:}b]/P$ divides K^1. The equation is in a hierarchy of order $m < n$ iff there is a divisor $Q \in \mathcal{H}_m$ of P such that P/Q divides K^1.*

Given $\mathcal{H}_n$ for all $n \in \mathbb{N}$, using this lemma, one can write down all approximately integrable equations (1) that are not in a lower hierarchy.

Example: Take $K^1 = K_1^{-1,2}, K_2^{1,0}$. With $r \neq -1$ the tuple $H = (1+y)(y-r)(yr-1), x(x+1+r)(rx+1+r)$ is a divisor of $\mathcal{G}_m[c{:}d] = \mathcal{G}_{1;m}^{-1,2}[c{:}d](1,y), \mathcal{G}_{2;m}^{1,0}[c{:}d](x,1)$ when $m \in \{3,5,7,\ldots\}$, $c/d = (1+r^m)/(1+r)^m$. It has maximal degree, $H = P \in \mathcal{H}_3$. Hence any third order system of this type is approximately integrable. To write down minimal conditions on K^1 such that the equation is not in a lower hierarchy one should look for $Q \in \mathcal{H}_{m<n}$ of maximal degree. The condition is: neither $(1+y), x$ nor $(y-r)(yr-1), (x+1+r)(rx+1+r)$ should divide K^1. Otherwise the equation would be in a second, or first order hierarchy respectively, or in a zeroth order hierarchy when both tuples divide K^1.

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References

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