Lax representation for integrable OΔEs

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We derive a Lax-representation for integrable maps (or OΔEs) obtained by travelling wave reductions from integrable PΔEs with Lax pairs.

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1. Outline

It is believed that all integrable systems possess a Lax representation. OΔEs can be obtained from PΔEs by travelling wave reductions. Suppose the Lax representation of the PΔE is known. Then the question arises: do integrable maps obtained in this way posses a Lax representation? If yes, how does one obtain it? This paper provides the answer to the above question positively for so called (q,p)-travelling wave reductions introduced in [1]. One of the Lax-matrices for the OΔE coincides with the monodromy matrix, which is obtained by taking a product of PΔE Lax-matrices along the (q,p)-staircase.

2. Lax representation

Consider an integrable PΔE on a two-dimensional lattice, namely $f_{l,m} = f(v_{l,m}, v_{l+1,m}, v_{l,m+1}, v_{l+1,m+1}; \alpha_i) = 0$, with $l, m \in \mathbb{Z}$ and α_i parameters. This equation has a Lax representation if there are matrices L, M, N depending on a spectral parameter k such that $L_{l,m}M_{l,m}^{-1} - M_{l+1,m}^{-1}L_{l,m+1} = f_{l,m}N_{l,m}$, in which $f_{l,m}$ does not depend on k, and $N_{l,m}$ is nonsingular on the equation[a]. Similarly, an OΔE $f_n = f(v_n, v_{n+1}, \ldots, v_{n+a}; \alpha_i) = 0$, with $n, a \in \mathbb{Z}$ and α_i parameters, has a Lax representation if there are

[a]Note that the right hand side vanishes for solutions of the equation and is set to 0 by many authors.

matrices $\mathcal{L}$, $\mathcal{M}$, $\mathcal{N}$ depending on a spectral parameter k such that $\mathcal{M}_n\mathcal{L}_n - \mathcal{L}_{n+1}\mathcal{M}_n = f_n\mathcal{N}_n$. Right-multiplying by $-\mathcal{M}_n^{-1}$ we obtain the invariance of $\mathcal{L}_n$, i.e., $\mathrm{tr}\mathcal{L}_{n+1} - \mathrm{tr}\mathcal{L}_n = f_n\Lambda_n$, where $\Lambda_n = -\mathrm{tr}\mathcal{N}_n\mathcal{M}_n^{-1}$. The coefficients in the expansion in powers of the spectral parameter of the trace of the monodromy matrix give integrals of the mapping. A PΔE can be reduced to an OΔE through a travelling wave reduction by the ansatz $v_n = v_{l,m}$ via the similarity variable $n = ql + pm$, where q, p are coprimes and l, m are the independent lattice variables. It is clear that this relationship induces the periodicity condition $v_{l+p,m-q} = v_{l,m}$, which allows us to solve the initial value problem on the OΔE. The *staircase method* provides a way of generating invariants for OΔEs obtained in this way. The monodromy matrix $\mathcal{L}_n$ is defined to be the ordered product of Lax-matrices along a standard staircase.[1] For that purpose it is useful to introduce the following

Definition 2.1. Given $p, q, l \in \mathbb{N}$, let Q_n^l be a matrix such that

$$Q_n^l = \prod_{j=0}^{\overset{\frown}{\lfloor (p-l-1)/q \rfloor}} L_{n+jq+l} \cdot M_{n+l}^{-1}, \quad \text{where} \quad \prod_{j=a}^{\overset{\frown}{b}} f_j := f_b \cdots f_a.$$

Now we can state our main result, explicit formulae for the Lax-matrices of (q,p)-reductions in terms of the PΔE Lax-matrices.

Theorem 2.1. *Given $q, p \in \mathbb{N}$ with $gcd(q,p) = 1$ and $1 < q \le p$, there are integers $s, s^{-1} \in [0, q)$ such that $p \equiv s \bmod q$ and $ss^{-1} \equiv 1 \bmod q$. Thus, the Lax representation for the OΔE arising through a (q,p)-travelling wave reduction is given by*

$$\mathcal{L}_n = M_n^{-1} \prod_{i=1}^{q} Q_n^{m_i} \cdot M_n, \qquad \mathcal{M}_n = M_{n+1}^{-1} \prod_{i=s^{-1}+1}^{q} Q_n^{m_i} \cdot M_n,$$

where $m_i = si \bmod q$ and $\mathcal{N}_n = \mathcal{M}_n L_n^{-1} N_n M_n \mathcal{L}_n$.

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References

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