






Letter

# Using discrete Darboux polynomials to detect and determine preserved measures and integrals of rational maps

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## Abstract

In this letter we propose a systematic approach for detecting and calculating integrals and preserved measures of rational maps. The approach is based on the use of cofactors and discrete Darboux polynomials and relies on the use of symbolic algebra tools. We show, in three examples, how to use this method to detect and determine integrals and preserved measures of the considered rational maps.

Keywords: first integrals, Darboux polynomials, rational maps

(Some figures may appear in colour only in the online journal)

## 1. Introduction

The search for integrals and preserved measures of ordinary differential equations (ODEs) has been at the forefront of mathematical physics since the time of Galileo and Newton.

In this letter our aim will be to develop an analogous theory for the (arguably more general) discrete-time case. This will lead to essentially linear algorithms for detecting and determining preserved measures and first and second integrals of (discrete) rational maps (both integrable and non-integrable). But before we consider the discrete case, let us look at the continuous case, i.e. ODEs.

Consider two polynomials  $P_1$  and  $P_2$ . Then  $I := P_1/P_2$  is a rational integral of the ODE  $\frac{dx}{dt} = f(\mathbf{x})$  if

$$\dot{P}_1 P_2 - P_1 \dot{P}_2 = 0$$

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along solutions of the ODE. Here  $\dot{c}$  denotes  $\frac{dc}{dt}$ . For a polynomial ODE, the problem of finding  $P_1$  and  $P_2$ , as posed, is bilinear in the coefficients of the polynomials  $P_1, P_2$ .

### 1.1. Darboux polynomials (ODE case)

A very nice introduction to Darboux polynomials for ODEs was given by Goriely [6]. Darboux polynomials were already studied by Darboux, Poincaré, Painlevé and others, see [6], and are also known by several other names, including ‘second integrals’ and ‘weak integrals’.

Let  $P(\mathbf{x})$  and  $C(\mathbf{x})$  be polynomials. Then  $P(\mathbf{x})$  is called a Darboux polynomial of the ODE  $\frac{dx}{dt} = f(\mathbf{x})$ , where  $f$  is polynomial, if

$$\dot{P}(\mathbf{x}) = C(\mathbf{x})P(\mathbf{x}),$$

along solutions of the ODE. Here  $C(\mathbf{x})$  is called the cofactor of  $P$ . Note that  $P(\mathbf{x}(0)) = 0$  implies  $P(\mathbf{x}(t)) = 0$  for all  $t$ . Hence the set  $P(\mathbf{x}) = 0$  is an invariant set in phase space.

Consider two Darboux polynomials with the same cofactor  $C$ :

$$\begin{aligned} \dot{P}_1 = CP_1 &\Rightarrow \frac{d}{dt} \left( \frac{P_1}{P_2} \right) = \frac{\dot{P}_1 P_2 - P_1 \dot{P}_2}{P_2^2} = \frac{CP_1 P_2 - P_1 CP_2}{P_2^2} = 0, \end{aligned} \quad (1)$$

i.e. the ratio of two Darboux polynomials with the same cofactor is a rational integral. The converse is also true. However, finding  $C, P_1$  and  $P_2$  involves one bilinear problem, plus one linear problem.

More generally,

$$\begin{aligned} \dot{P}_1 = C_1 P_1 \\ \dot{P}_2 = C_2 P_2 \end{aligned} \Rightarrow \frac{d}{dt} (P_1 P_2) = \dot{P}_1 P_2 + P_1 \dot{P}_2 = (C_1 + C_2) P_1 P_2. \quad (2)$$

### 1.2. Discrete Darboux polynomials (mapping case)

Instead of polynomial ODEs  $\frac{dx}{dt} = f(\mathbf{x})$ , we now consider rational maps  $\mathbf{x}_{n+1} = \phi(\mathbf{x}_n)$  (see [4, 5]). Then we define  $P(\mathbf{x})$  to be a discrete Darboux polynomial of the rational map  $\mathbf{x}_{n+1} = \phi(\mathbf{x}_n)$  if

$$P(\mathbf{x}_{n+1}) = C(\mathbf{x}_n)P(\mathbf{x}_n), \quad (3)$$

where the cofactor  $C$  is now a rational function whose form will be presented in section 1.3. We use the shorthand notation

$$P' = CP.$$

Note that, similarly to the continuous case,  $P(\mathbf{x}) = 0$  is an invariant set in phase space.

Now consider again two discrete Darboux polynomials  $P_1$  and  $P_2$  with the same cofactor  $C$ :

$$\begin{aligned} P'_1 = CP_1 \\ P'_2 = CP_2 \end{aligned} \Rightarrow \frac{P'_1}{P'_2} = \frac{P_1}{P_2},$$

i.e. the ratio of the two discrete Darboux polynomials with the same cofactors is again an integral (and the converse is also true). More generally

$$\begin{aligned} P'_1 = C_1 P_1 \\ P'_2 = C_2 P_2 \end{aligned} \Rightarrow (P_1 P_2)' = C_1 C_2 (P_1 P_2).$$

How is all this going to help us find integrals of a given map? The answer comes in two parts:

1. In the discrete case we use a non-trivial ansatz for the cofactors  $C(\mathbf{x})$ .
2. In the discrete case the cofactor of the product is the *product* of the cofactors.  
 In the continuous case the cofactor of the product is the *sum* of the cofactors.

The latter point is crucial: it means that in the discrete case we can use the fact that the factorization of the cofactor  $C$  is unique. By contrast, in the ODE case we have addition, where splitting into summands is not unique.

### 1.3. Ansatz

Given a rational map  $\phi$  with common denominator

$$D(\mathbf{x}) = \prod_{j=1}^k D_j^{d_j}(\mathbf{x})$$

and Jacobian determinant

$$J(\mathbf{x}) = \frac{\prod_{i=1}^l K_i^{b_i}(\mathbf{x})}{\prod_{j=1}^k D_j^{e_j}(\mathbf{x})},$$

where the  $K_i(\mathbf{x})$  are  $l$  distinct factors<sup>5</sup>, we try all cofactors (up to a certain polynomial degree  $d$ ) of the form

$$C(\mathbf{x}) = \pm \frac{\prod_{i=1}^l K_i^{f_i}(\mathbf{x})}{\prod_{j=1}^k D_j^{g_j}(\mathbf{x})}, \tag{4}$$

where  $f_j, g_j \in \mathbb{N}_0$ .

Comments:

1. For the finite number of cofactors of degree  $< d$ , we only need to solve the linear problem (3) to determine the corresponding Darboux polynomial  $P$  (up to a chosen degree).
2. If  $C(\mathbf{x}) = J(\mathbf{x})$ , the corresponding Darboux polynomials are (inverse) densities of preserved measures.
3. Note that we include  $\pm 1, \pm J$  in the list of potential cofactors to be examined.

The above approach is particularly useful for maps for which no other systematic method is known to derive integrals, prime examples being Kahan maps [2, 15] and dual maps [16]. In the remainder of this Letter we therefore study two maps that arise as Kahan–Hirota–Kimura (KHK) discretizations of quadratic ODEs (in section 2.1, resp. section 3) [9, 11], plus a third map that arises as the reduction of the dual AKP equation (in section 2.2) [19].

For any given cofactor, of the form (4), the equation (3) gives rise to a system of equations that are linear in the coefficients,  $c_i$ , of the discrete Darboux polynomial  $P$ , and may also depend on the parameters  $\alpha_j$  in the mapping through both  $x_{n+1}$  and  $C(x_n)$  in (3). One can either treat the resulting system as a linear system to determine the  $c_i$  (as we do e.g. in section 2.2) or solve the (nonlinear) system to simultaneously determine the  $c_i$  and detect the  $\alpha_j$  (as we do in section 3). In this paper we use standard Maple routines to perform both kinds of computations.

<sup>5</sup>Factorisation is carried out over the field implied by the coefficients of the map  $\phi$ .

## 2. Determining preserved measures and first and second integrals of rational maps

In this section, we study two examples of discrete maps for which we use the ansatz of section 1.3 to find their preserved integrals and measures.

### 2.1. Determining measures and integrals of a 2D Lotka–Volterra system

In this sub-section we study the following two-dimensional ODE as an example:

$$\begin{aligned} \frac{dx}{dt} &= x(x + 6y - 3) \\ \frac{dy}{dt} &= y(-3y - 2x + 3). \end{aligned} \tag{5}$$

The KHK discretization of (5) reads (see [2, 3, 7, 9–11, 15])

$$\begin{aligned} x' &= \frac{x(1 + h(x + 6y - 3) + \frac{h^2}{4}(9 - 6x))}{D(x)} \\ y' &= \frac{y(1 + h(3 - 2x - 3y) + \frac{9h^2}{4}(1 - 2y))}{D(x)} \end{aligned} \tag{6}$$

where the common denominator  $D(\mathbf{x})$  of the map is given by

$$D(\mathbf{x}) := 1 - \frac{h^2}{4}(9 - 12x - 36y + 4x^2 + 12xy + 36y^2). \tag{7}$$

The Jacobian determinant  $J(\mathbf{x})$  of the mapping (6) is

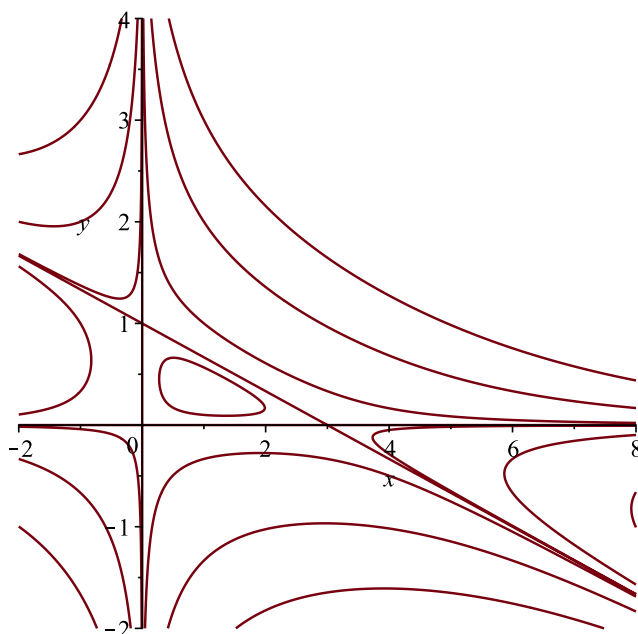
$$J(\mathbf{x}) = \frac{K_1(\mathbf{x})K_2(\mathbf{x})K_3(\mathbf{x})}{D^3(\mathbf{x})} \tag{8}$$

where

$$\begin{aligned} K_1 &= 1 + h(x - 3y) - \frac{3}{4}h^2(3 - 2x - 6y) \\ K_2 &= 1 + h(x + 6y - 3) + \frac{3}{4}h^2(3 - 2x) \\ K_3 &= 1 + h(3 - 2x - 3y) + \frac{9}{4}h^2(1 - 2y). \end{aligned} \tag{9}$$

We have used cofactors  $C_1 = \frac{K_1}{D}$ ,  $C_2 = \frac{K_2}{D}$ ,  $C_3 = \frac{K_3}{D}$ ,  $C_4 = J$  to find the corresponding discrete Darboux polynomials for the map (6):

$$\begin{aligned} p_{1,1} &= x + 3y - 3 \\ p_{2,1} &= x \\ p_{3,1} &= y \\ p_{4,1} &= xy(x + 3y - 3) \\ p_{4,2} &= 1 - \frac{h^2}{4}(9 - 12x - 36y + 4x^2 + 12xy + 36y^2). \end{aligned} \tag{10}$$



**Figure 1.** Phase plot for map (6), for  $h = \frac{1}{17}$ .

Here, and in section 2.2,  $p_{i,j}$  denotes the  $j$ th Darboux polynomial corresponding to the cofactor  $C_i$ .

A phase plot for the map (6), clearly exhibiting the linear Darboux polynomials  $p_{1,1}$ ,  $p_{2,1}$ , and  $p_{3,1}$ , is given in figure 1.

It follows that the map (6) preserves the integral

$$\tilde{I}(\mathbf{x}) = \frac{xy(x + 3y - 3)}{1 - \frac{h^2}{4}(9 - 12x - 36y + 4x^2 + 12xy + 36y^2)} \tag{11}$$

and the measure

$$\frac{dxdy}{1 - \frac{h^2}{4}(9 - 12x - 36y + 4x^2 + 12xy + 36y^2)}. \tag{12}$$

Taking the continuum limit  $h \rightarrow 0$ , we obtain the cofactors  $\tilde{C}_1 = x - 3y$ ,  $\tilde{C}_2 = x + 6y - 3$ ,  $\tilde{C}_3 = 3 - 2x - 3y$ ,  $\tilde{C}_4 = 0$ , and the corresponding Darboux polynomials

$$p_{1,1} = x + 3y - 3, \quad p_{2,1} = x, \quad p_{3,1} = y, \quad p_{4,1} = xy(x + 3y - 3), \quad p_{4,2} = 1$$

It follows that the ODE (5) preserves the integral

$$I(\mathbf{x}) = xy(x + 3y - 3) \tag{13}$$

and the measure

$$dxdy. \tag{14}$$

It thus turns out that our original ODE (5) is Hamiltonian, with  $H(\mathbf{x}) = xy(x + 3y - 3)$ .

Interpreted conversely, one can say that the KHK discretization (6) preserves the three affine Darboux polynomials of the ODE (5), as well as the modified integral (11) and the modified density (12). These results are no coincidences.

Indeed, the preservation of the three affine Darboux polynomials is the consequence of the following theorem (whose proof is presented in appendix A).

**Theorem 1.** *The KHK discretization preserves all affine Darboux polynomials of a given quadratic ODE.*

But there is more to be said. Looking at equations (8)–(10), one notices that to each affine Darboux polynomial ( $x + 3y - 3$ , resp.  $x$ , resp.  $y$ ) there is a corresponding factor in the numerator of  $J(\mathbf{x})$  (viz.  $K_1$ , resp.  $K_2$ , resp.  $K_3$ ). This again is no coincidence. Indeed, the latter fact is a consequence of the following Theorem (whose proof is presented in appendix B).

**Theorem 2.** *Let  $\frac{dx}{dt} = f(\mathbf{x})$  be a quadratic differential equation in 1, 2, 3 or 4 dimensions, that possesses an affine Darboux polynomial  $P(\mathbf{x})$ . Then the numerator of the discrete cofactor of  $P$  in the corresponding Kahan map  $\mathbf{x}' = \phi_h(\mathbf{x})$  is a factor of the numerator of the Jacobian determinant of  $\phi_h$ .*

Theorems 1 and 2 are a very significant step towards the full resolution of the open problem posed in 2002 in [13]: ‘How does one preserve more than  $n - 1$  integrals and weak integrals (of an  $n$ -dimensional vector field)?’

The preservation of the modified integral and measure is an example of a general result in [2] giving a modified integral for all systems with a cubic Hamiltonian in any dimension.

## 2.2. Determining measures and integrals of a 9D dual AKP-reduction

In [19], Van der Kamp, Quispel and Zhang derived a dual AKP equation. Here we consider its (1, 2, 4)-reduction, given by the 9D map:

$$\begin{cases} x'_i = x_{i+1} & 1 \leq i \leq 8 \\ x'_9 = \frac{\alpha_1}{\alpha_3} \left( \frac{x_7^2}{x_4} - \frac{x_4 x_6 x_8}{x_3 x_5} \right) + \frac{\alpha_2}{\alpha_3} \left( \frac{x_5 x_8}{x_3} - \frac{x_6^2 x_7}{x_4 x_5} \right) + \frac{x_1 x_6 x_7 x_8}{x_4 x_3 x_5} + \frac{\alpha_4}{\alpha_3} \left( \frac{x_2 x_7 x_8}{x_4 x_3} - \frac{x_6 x_9}{x_5} \right), \end{cases}$$

where  $\alpha_1, \alpha_2, \alpha_3, \alpha_4$  are parameters. Its Jacobian determinant is  $J(\mathbf{x}) = \frac{x_6 x_7 x_8}{x_4 x_3 x_5}$ . We have used cofactors

$$C_1(\mathbf{x}) = \frac{x_7}{x_4}, \quad C_2(\mathbf{x}) = \frac{x_8}{x_3}, \quad C_3(\mathbf{x}) = \frac{x_6 x_8}{x_3 x_5}$$

to find Darboux polynomials. Seven of these are listed below.

$$\begin{aligned} p_{1,1}(\mathbf{x}) &= \alpha_1 x_3 x_7 x_5 + (-x_3 x_6^2 - x_4^2 x_7) \alpha_2 + (x_1 x_6 x_8 + x_2 x_4 x_9) \alpha_3 + \alpha_4 x_2 x_8 x_5 \\ p_{1,2}(\mathbf{x}) &= x_4 x_6 x_5 \end{aligned}$$

$$\begin{aligned} p_{2,1}(\mathbf{x}) &= -x_4^2 x_6^2 \alpha_1 x_5 + x_5^3 a_2 x_4 x_6 + (x_1 x_5 x_6^2 x_7 + x_2 x_3 x_6 x_7^2 + x_3^2 x_4 x_7 x_8 + x_3 x_4^2 x_5 x_9) \alpha_3 \\ &\quad + (x_2 x_5^2 x_6 x_7 + x_3^2 x_6^2 x_7 + x_3 x_4^2 x_7^2 + x_3 x_4 x_5^2 x_8) \alpha_4 \end{aligned}$$

$$\begin{aligned} p_{2,2}(\mathbf{x}) &= (-x_3 x_4 x_6^3 - x_3 x_5^3 x_7 - x_4^3 x_6 x_7) \alpha_1 + (x_3 x_5^2 x_6^2 + x_4^2 x_5^2 x_7) \alpha_2 \\ &\quad + (x_1 x_4 x_6 x_7^2 + x_2 x_3 x_5 x_7 x_8 + x_3^2 x_4 x_6 x_9) \alpha_3 + (x_2 x_4 x_5 x_7^2 + x_3^2 x_5 x_6 x_8) \alpha_4 \end{aligned}$$

$$p_{2,3}(\mathbf{x}) = x_3x_7x_4x_5x_6$$

$$p_{3,1}(\mathbf{x}) = (-x_2x_3x_5x_6x_7^2 + x_2x_4^2x_6^2x_8 - x_3^2x_4x_5x_7x_8) \alpha_1^2 + (x_2x_3x_6^3x_7 - x_2x_4x_5^2x_6x_8 + x_3x_4^3x_7x_8) \alpha_1\alpha_2$$

$$+ (-x_1x_2x_6^2x_7x_8 - x_1x_3x_5^2x_7x_9 - x_2x_3x_4^2x_8x_9) \alpha_1\alpha_3 + (-x_1x_3x_5x_7^3 + x_1x_4^2x_6x_7x_8 - x_2^2x_5x_6x_7x_8$$

$$- x_2x_3x_4x_5x_8^2 + x_2x_3x_4x_6^2x_9 - x_3^3x_5x_7x_9) \alpha_1\alpha_4 + x_3^2x_7^2\alpha_2^2x_4x_6 - x_3x_7(x_1x_4x_7x_8 + x_2x_3x_6x_9) \alpha_2\alpha_3$$

$$+ (x_1x_3x_6^2x_7^2 - x_1x_4x_5^2x_7x_8 - x_2x_3x_5^2x_6x_9 + x_3^2x_4^2x_7x_9) \alpha_2\alpha_4 + \alpha_3^2x_1x_2x_8x_9x_7x_3 + (-x_1^2x_6x_7^2x_8$$

$$- x_2x_3^2x_4x_9^2) \alpha_4\alpha_3 + (-x_1x_2x_5x_7^2x_8 + x_1x_3x_4x_6x_7x_9 - x_2x_3^2x_5x_8x_9) \alpha_4^2$$

$$p_{3,2}(\mathbf{x}) = x_3x_4x_5^2x_6x_7.$$

The four integrals

$$k_1(\mathbf{x}) = \frac{p_{1,1}}{p_{1,2}}, \quad k_2(\mathbf{x}) = \frac{p_{2,1}}{p_{2,3}}, \quad k_3(\mathbf{x}) = \frac{p_{2,2}}{p_{2,3}}, \quad k_4(\mathbf{x}) = \frac{p_{3,1}}{p_{3,2}}$$

are functionally independent.

Since  $C_1(\mathbf{x})C_3(\mathbf{x}) = J(\mathbf{x})$ , it follows that the map also preserves the measure with density  $p_{1,2}(\mathbf{x})^{-1}p_{3,2}(\mathbf{x})^{-1} = (x_3x_4^2x_5^3x_6^2x_7)^{-1}$ .

### 3. Detecting preserved measures and first and second integrals of rational maps

In this section we consider the following three-dimensional ODE as an example:

$$\frac{dx}{dt} = x(y - \alpha_1z), \quad \frac{dy}{dt} = y(\alpha_2z - x), \quad \frac{dz}{dt} = z(\alpha_1x - \alpha_2y), \quad (15)$$

where  $\alpha_2$  and  $\alpha_1$  are arbitrary parameters.

Applying the Kahan–Hirota–Kimura discretization to (15), we obtain

$$\frac{x' - x}{h} = \frac{x'(y - \alpha_1z) + x(y' - \alpha_1z')}{2}$$

$$\frac{y' - y}{h} = \frac{y'(\alpha_2z - x) + y(\alpha_2z' - x')}{2}$$

$$\frac{z' - z}{h} = \frac{z'(\alpha_1x - \alpha_2y) + z(\alpha_1x' - \alpha_2y')}{2}. \quad (16)$$

Solving equation (16) for  $x'$ ,  $y'$ , and  $z'$  we obtain the (rational) Kahan map discretizing (15). Using the Jacobian determinant  $J(\mathbf{x})$  of the Kahan map as cofactor, our algorithm finds that for all  $(\alpha_1, \alpha_2)$ , the map preserves the measure  $\frac{dx dy dz}{xyz}$  and the first integral  $x + y + z$ .

Moreover, the algorithm also detects the following special values of the parameters  $(\alpha_1, \alpha_2)$  where the map preserves an additional integral, and outputs the formula for the integral (see table 1).

**Table 1.** Integrable parameter values and corresponding functionally independent additional first integrals detected by our algorithm.

$(\alpha_1, \alpha_2)$	Additional first integral
$(-1, 0)$	$y/z$
$(1, 0)$	$yz/(1 - \frac{h^2}{4}x^2)$
$(0, 1)$	$xz/(1 - \frac{h^2}{4}y^2)$
$(0, -1)$	$z/x$
$(1, 1)$	$xyz/(1 - \frac{h^2}{4}(x^2 + y^2 + z^2 - 2xy - 2xz - 2yz))$
$(1, -1)$	$x/yz$
$(-1, -1)$	$z/xy$
$(-1, 1)$	$y/xz$

#### 4. Concluding remarks

In this letter we have presented a method for detecting and determining first and second integrals of rational maps. There are in the literature several other methods for *determining* first and second integrals of discrete systems, see [4, 5, 8, 14, 18] and references therein. There are also in the literature several other methods for *detecting* first and second integrals of discrete systems, see [1, 7, 17] and references therein.

However, to our knowledge none of the above combine all the following properties of the method presented in this letter:

1. It is algorithmic, and requires no other input than the rational map in question. At heart the algorithm is linear and, to some extent apart from birationality, requires no knowledge about the map (such as symplecticity, measure preservation, time-reversal symmetry, integrability, Lax pairs, etc) on the part of the user.
2. Up to a certain prescribed degree, it determines and outputs all
  - (a) rational first integrals
  - (b) polynomial second integrals
  - (c) preserved measures of the form  $P(x)dx$  or  $\frac{dx}{P(x)}$ , where  $P$  is a polynomial whose cofactors are of the form (4).
3. It can detect special parameter values where additional first and/or second integrals and/or preserved measures exist, and output those integrals, preserved measures and parameter values.
4. It works for both integrable and non-integrable cases, in arbitrary dimension.
5. It allows one to take the continuum limit, if appropriate.

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### Appendix A. Proof of theorem 1

For quadratic vector fields  $f(\mathbf{x})$ , Kahan’s method is equivalent to the B-series method

$$\mathbf{x}' = \phi(\mathbf{x}) := \mathbf{x} + h \left( \mathbf{I} - \frac{1}{2}hf'(\mathbf{x}) \right)^{-1} f(\mathbf{x}) \tag{A.1}$$

where  $f'$  is the Jacobian matrix of  $f$  [2].

**Lemma A.1.** *The B-series method (A.1) preserves all affine Darboux polynomials of any polynomial ODE.*

**Proof.** Since B-series methods commute with affine transformations [12], we may assume that the Darboux polynomial to be preserved equals  $x_1$ , i.e.

$$\frac{dx_1}{dt} = x_1 \cdot C(\mathbf{x}), \tag{A.2}$$

where the cofactor  $C$  is polynomial. Using Cramer’s rule it follows, using (A.1), that

$$\frac{x'_1 - x_1}{h} = \left( M_{11}f_1 + \frac{h}{2} \sum_{j=2}^n M_{1j}f_j \frac{\partial f_1}{\partial x_j} \right) / \det \left( \mathbf{I} - \frac{h}{2}f' \right) \tag{A.3}$$

where the  $M_i$  are (determinantal) polynomials depending on  $h$  and  $x$ . Substituting (A.2) in (A.3), we obtain

$$x'_1 = x_1 \left( 1 + hCM_1 + \frac{h^2}{2} \sum_{j=2}^n M_{1j}f_j \frac{\partial C}{\partial x_j} \right) / \det \left( \mathbf{I} - \frac{h}{2}f' \right).$$

This shows that the discrete cofactor is of the form (4), and concludes the proof of lemma A.1. □

Since Kahan’s method is equivalent to the B-series method (A.1) for quadratic vector fields, theorem 1 follows as a corollary.

### Appendix B. Proof of theorem 2

In order to prove theorem 2, we first prove the following:

**Lemma B.1.** *Let  $\phi_h$  be given by<sup>6</sup>*

$$x'_1 = x_1 \frac{N_1(x, h)}{D(x, h)}, \text{ with } N_1 \text{ and } D \text{ coprime,} \tag{B.1}$$

$$x'_i = \frac{N_i(x, h)}{D(x, h)}, \quad i = 2, \dots, n. \tag{B.2}$$

Also assume that the Jacobian determinant  $J$  of  $\phi_h$  satisfies

$$J = \frac{D(x', -h)}{D(x, h)}, \tag{B.3}$$

<sup>6</sup>Capitalised functions are polynomial in their arguments.

and let

$$\phi_h \circ \phi_{-h} = id, \tag{B.4}$$

i.e. the system is invariant under

$$x \leftrightarrow x'. \tag{B.5}$$

Then  $N_1$  divides the numerator of the Jacobian determinant  $J$ .

**Proof.** Applying (B.5) to (B.1), we get

$$x_1 = x'_1 \frac{N_1(x', -h)}{D(x', -h)} = x_1 \frac{N_1(x, h)}{D(x, h)} \frac{N_1(x', -h)}{D(x', -h)}.$$

Hence

$$N_1(x, h)N_1(x', -h) = D(x, h)D(x', -h). \tag{B.6}$$

From (B.1) and (B.2) it follows that  $D(x', -h)$  can be written as

$$D(x', -h) = \frac{L(x, h)}{D^k(x, h)} \tag{B.7}$$

for some  $k \in \mathbb{N}_0$ . It follows from (B.3) that

$$J = \frac{L(x, h)}{D^{k+1}(x, h)} =: \frac{\bar{L}(x, h)}{\bar{D}(x, h)}, \tag{B.8}$$

where  $\bar{L}$  and  $\bar{D}$  are coprime. Using (B.7) and (B.8) in (B.6) we get

$$\bar{D}(x, h)N_1(x, h)N_1(x', -h) = D^2(x, h)\bar{L}(x, h).$$

From (B.1) and (B.2) it also follows that  $N_1(x', -h)$  can be written as

$$N_1(x', -h) = \frac{M(x, h)}{D^m(x, h)}, \quad \text{where, for Kahan's method } m = 1.$$

Hence

$$\bar{D}(x, h)N_1(x, h)M(x, h) = D^{2+m}(x, h)\bar{L}(x, h).$$

Hence  $N_1 | \bar{L}$ , i.e. the numerator of the cofactor of  $x_1$  divides the numerator of the Jacobian determinant  $J$ . □

What remains to prove is that in dimensions 1, 2, 3, and 4 the Kahan map satisfies theorem 2. The first step towards proving this is to use the fact that if  $\phi_h$  is a Kahan map, then we may assume w.l.o.g. that the affine Darboux polynomial is  $x_1$ , and hence the Kahan map has the form (B.1) and (B.2) and its Jacobian determinant satisfies (B.3) and (B.4) [2].

The only thing we still need to show is that  $N_1$  and  $D$  are coprime, where  $D(x, h) = \det(I - \frac{1}{2}hf'(x))$ . This we have done explicitly, using Maple for the most general quadratic ODEs with Darboux polynomial  $x_1$ .

The 5D case being outside the capabilities of the supercomputer at our disposal, we have performed 1000, resp. 50, resp. 6 computations similar to the ones above in 5D, resp. 6D, resp. 7D, but with randomly chosen integer values between  $-100$  and  $100$  for the parameters instead of symbolic ones.

In all computations,  $N_1$  and  $D$  are coprime.

Based on theorem 2 and on the computations mentioned above, we make the following:

**Conjecture.** Theorem 2 (which we proved in dimensions 1, 2, 3, and 4) remains true in any dimension.

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