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Twisted reductions of integrable lattice equations, and their Lax representations

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Abstract
It is well known that from two-dimensional lattice equations one can derive one-dimensional lattice equations by imposing periodicity in some direction. In this paper we generalize the periodicity condition by adding a symmetry transformation and apply this idea to autonomous and non-autonomous lattice equations. As results of this approach, we obtain new reductions of the discrete potential Korteweg–de Vries (KdV) equation, discrete modified KdV equation and the discrete Schwarzian KdV equation. We will also describe a direct method for obtaining Lax representations for the reduced equations.

Keywords: integrable, lattice equations, reductions, discrete Painleve
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1. Introduction

A key property of integrable partial differential equations (PDEs) is the existence of multisoliton solutions describing the elastic scattering between the solitons. The single soliton can also be called a travelling wave solution, as its form is unchanged after some time, up to translation [1]. Such invariances are generalized and formalized in the symmetry approach [5, 34], where one uses symmetries of the original equation to derive an additional equation, the similarity constraint, which is compatible with the original equation. One can then use this constraint equation to reduce the original integrable PDE to an integrable ordinary differential equation (ODE). For example, in the case of the Korteweg–de Vries (KdV) equation

$$\partial_t u + \partial_x^3 u + u \partial_x u = 0, \tag{1.1}$$

the constraint $v\partial_x u + \partial_t u = 0$ leads to the travelling wave ansatz $u = f(x - vt)$ and elliptic equation for f, while the similarity constraint $2u + x\partial_x u + 3t\partial_t u = 0$ leads to the similarity ansatz $u = t^{-2/3}\phi(z)$, $z = x/(t^{1/3})$ and then to an equation for ϕ that can be transformed, by letting $\phi = \partial_z y - y^2/6$, to

$$\partial_z^2 y = \frac{y^3}{18} + \frac{yz}{3} + \alpha, \tag{1.2}$$

(where α is an integration constant) which is the second Painlevé equation (see [34] p 195). Given a constraint, a method for obtaining a Lax pair for the reduced equation was given in [13]. For further applications of symmetries of PDEs in mathematical physics see [41, 42] and references therein.

Integrable PDEs, or lattice equations, can be seen as discrete analogues of integrable PDEs, and they have been shown to possess many of the same characteristics as their continuous analogues, such as Lax representations [30, 31, 45], bilinear structures and N-soliton solutions [9–11, 22–24]. Furthermore, several approaches have been developed to reduce PDEs to ordinary difference equations (ODEs) [20, 29, 39, 44, 55–58]. Reductions of the kind of those presented here are obtained in [27, 43] using Lie group techniques in the case of differential–difference equations.

We consider equations defined on the Cartesian two-dimensional lattice. In this context a particularly interesting set of equations is given by the form

$$Q(w_{l,m}, w_{l+1,m}, w_{l,m+1}, w_{l+1,m+1}; \alpha_l, \beta_m) = 0, \ \forall l, m, \tag{1.3}$$

where the subscripts, l, m, indicate a point in the Cartesian two-dimensional lattice on which the dependent variable w is defined, and α_l and β_m are lattice parameters associated with the horizontal and vertical edges, respectively[4]. Such equations are often called quad equations, because the equation connects values of w given at the corners of an elementary quadrilateral of the lattice. If the parameters α_l and β_m do not depend on the coordinates l, m, respectively, then the equation is said to be autonomous. We assume also that equation (1.3) is multilinear so that we can solve for any particular corner value in terms of the other three.

For quadrilateral equations one definition of integrability is by 'multidimensional consistency' [32, 33]. This has turned out to be a very effective definition, and in its three-dimensional version (consistency-around-a-cube, CAC) it has led (under some mild additional assumptions) to a classification of scalar integrable quadrilateral equations [3,4]. Our examples have been chosen from this class of equations. One very important consequence of the CAC property is that it immediately provides a Lax pair [28], which is a system of linear difference equations whose consistency is equivalent to the equation (1.3).

One may consider the analogue of a travelling wave solution to be a solution on the lattice admitting the constraint[5]

$$w_{l+s_1,m+s_2} = w_{l,m}, \tag{1.4}$$

leading to what is known as an (s_1, s_2)-reduction [56]. In order to construct consistent evolution we have to consider initial values satisfying this constraint and make sure that the evolution does not break the constraint.

In a similar manner to the continuous case, where reductions of PDEs lead to interesting ODEs, many authors have identified reductions given by (1.4) with interesting ODEs such as discrete analogues of elliptic functions, known as QRT maps [29], discrete Painlevé equations [15, 17, 37, 38, 48] and many higher dimensional mappings [20, 39, 44, 55, 58].

[4] We note that reductions from PDEs to ODEs have also been derived for equations depending on more points of the lattice, as well as for systems of equations [56, 58].

[5] Another type of reduction, via a nonlinear similarity constraint, was given in [29].

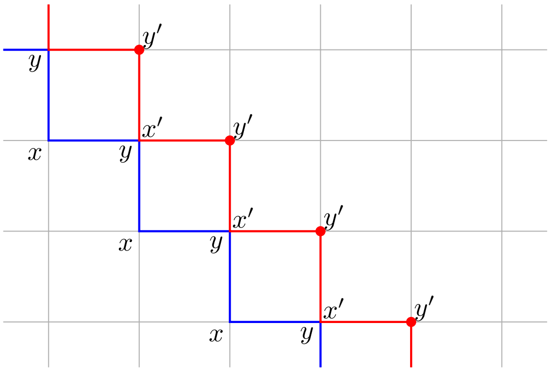

Figure 1. Labelling of variables for the (1,-1)-reduction of the lattice.

Of particular interest to this study are QRT maps and discrete Painlevé equations, which are both classes of integrable second order nonlinear difference equations. The QRT maps are autonomous mappings that preserve a biquadratic invariant [46, 47] whereas discrete Painlevé equations are integrable non-autonomous difference equations admitting the classical Painlevé equations as continuum limits [51] and also QRT maps as autonomous limits [49]. For example, two discretizations of (1.2) are

$$y_{n+1} + y_{n-1} = \frac{y_n(hn+a)+b}{1-y_n^2},$$

$$y_{n+1}y_ny_{n-1} = \frac{aq^n y_n(y_n - q^n)}{y_n - 1},$$

which are called multiplicative and additive difference equations in accordance with their dependence on n [51]. Their autonomous limits, when $h \to 0$ and $q \to 1$, respectively, are QRT maps [46, 47].

Let us consider the simplest (non-trivial) case of a periodic reduction, determined by the constraint $w_{l+1,m-1} = w_{l,m}$. We can then give the initial values on the blue staircase given in figure 1. In this case only two initial values are needed, x and y. Solving for $w_{l+1,m+1}$ from (1.3) we obtain

$$w_{l+1,m+1} = f(w_{l,m}, w_{l+1,m}, w_{l,m+1}; \alpha, \beta)$$

for some rational function f (here we assume the parameters α, β are constants). From figure 1 we then find that the initial values on the staircase evolve by the two-dimensional map

$$x' = y, \quad y' = f(x, y, y; \alpha, \beta),$$

and that in particular the periodicity is preserved. This result can also be written as a second order ordinary difference equation of the form

$$x_{n+2} = f(x_n, x_{n+1}, x_{n+1}; \alpha, \beta).$$

What is important is that if the original PDE (1.3) is integrable and has a Lax pair then it is possible to construct a Lax pair for the resulting ordinary difference equation, which therefore is integrable as well.

Recently three of the authors presented a direct method for obtaining the Lax representations of equations arising as periodic reductions of non-autonomous lattice equations [37, 38], which can be considered as the discretization of the method given in [13].

In this paper we consider the generalization of (1.4) in the form

$$w_{l+s_1,m+s_2} = T(w_{l,m}), \tag{1.5}$$

where the transformation T (which we call the 'twist') is fractional linear, which is also known as a homographic transformation [12]. In the example discussed above we would impose $w_{l+1,m-1} = T(w_{l,m})$ and start with a sequence of initial values of the form

$$\ldots, T^{-2}(y), T^{-1}(x), T^{-1}(y), x, y, T(x), T(y), \ldots$$

and after one step of evolution the new values should be similarly related, i.e.

$$\ldots, T^{-2}(y'), T^{-1}(x'), T^{-1}(y'), x', y', T(x'), T(y'), \ldots$$

Thus, on the kth step of the staircase we would get the evolution

$$T^k(x') = T^k(y), \qquad T^k(y') = f(T^{k+1}(x), T^{k+1}(y), T^k(y)),$$

But since $y' = f(T(x), T(y), y)$ this makes sense only if

$$T^k(f(T(x), T(y), y)) = f(T^{k+1}(x), T^{k+1}(y), T^k(y)),$$

in other words, equation (1.3) must be invariant under the transformation T, i.e.

$$Q(\{T(w_{l,m})\}; \alpha_l, \beta_m) \propto Q(\{w_{l,m}\}; \alpha_l, \beta_m).$$

The main result of the paper is a method for calculating Lax representations for these reductions, even in the non-autonomous case.

The paper is organized as follows: first in section 2 we review the reduction method for the $s_1 = 2, s_2 = 1$ reduction and then discuss the possible non-autonomous parameters of the equation. We distinguish the following cases, based on how the lattice parameters α_l and β_m vary:

- the autonomous case, where the parameters are constant;
- the simply non-autonomous case, where the parameters depend only explicitly on the lattice position; and
- the fully non-autonomous case, where the parameters also depend on additional constants, which are not left invariant under a lattice shift.

Each of these three cases exists in a twisted and a non-twisted version. We will review these parameter choices in more depth in section 2.

In section 3 we present the general method for constructing the Lax matrices. To illustrate our method we then perform (2, 1)-reductions of three archetypical equations with distinct twists. The first equation of the form (1.3), considered in section 4, will be the discrete modified Korteweg–de Vries (dmKdV) equation , also called $H3_{\delta=0}$, where

$$Q_{H3_{\delta=0}} = \alpha_l(w_{l,m}w_{l+1,m} - w_{l,m+1}w_{l+1,m+1}) - \beta_m(w_{l,m}w_{l,m+1} - w_{l+1,m}w_{l+1,m+1}), \tag{1.6}$$

with twist $T_1 : w \to w\lambda$. Here we will review the non-twisted autonomous case, the twisted autonomous case, the twisted simply non-autonomous case and the twisted fully non-autonomous case. We will also briefly study a second twist, $T_2 : w \to \frac{\lambda}{w}$.

The second equation, considered in section 5, will be the lattice potential KdV equation, or $H1$, where

$$Q_{H1} = (w_{l,m} - w_{l+1,m+1})(w_{l+1,m} - w_{l,m+1}) - \alpha_l + \beta_m, \tag{1.7}$$

with twists $T_1 : w \to w+\lambda$ and $T_2 : w \to \lambda - w$. Here we will consider the twisted autonomous case, the twisted simply non-autonomous case and the twisted fully non-autonomous case. In section 6, we will consider the lattice Schwarzian KdV equation, or $Q1_{\delta=0}$, with

$$Q_{Q1_{\delta=0}} = \alpha_l[(w_{l,m} - w_{l,m+1})(w_{l+1,m} - w_{l+1,m+1})] - \beta_m[(w_{l,m} - w_{l+1,m})(w_{l,m+1} - w_{l+1,m+1})], \tag{1.8}$$

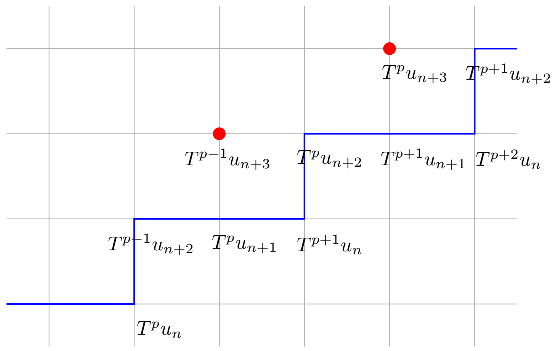

Figure 2. Labelling of variables for the (2,1)-reduction of the lattice with respect to (2.2).

where we will consider the twisted autonomous case and the twisted fully non-autonomous case. The twist will be an arbitrary Möbius transformation.

Finally, in section 7 we will consider the (2,2)-reduction of (1.8), and obtain the full parameter q-P$_{VI}$. In section 8 we treat the general (s_1, s_2)-reduction, and provide a list of twists for ABS-equations [3, 4].

While this paper was being edited, the preprint [18] appeared on the arXiv, which presents a twisted version of the approach in [48].

2. Symmetry invariance

For pedagogical reasons we specialize our reduction, given by (1.5), to one of the simplest possible cases; where $s_1 = 2$ and $s_2 = 1$. In contrast to the case $s_1 = s_2 = 1$, in this case there is a difference between the simply non-autonomous case and the fully non-autonomous case. In this special case, our reduction may be specified by introducing two variables,

$$n = 2m - l, \qquad p = l - m. \tag{2.1}$$

We label the variables of the reduction in terms of n and p by specifying

$$w_{l,m} \mapsto T^{l-m} u_{2m-l} = T^p u_n. \tag{2.2}$$

This extends the labelling of [37] to accommodate for the twist. With this constraint, it is sufficient to specify just three initial conditions. Their values, and the values obtained from the similarity constraint, (1.5), form a staircase which determines a solution on all of $\mathbb{Z}^2$. A small portion of the staircase in $\mathbb{Z}^2$ has been depicted in figure 2.

The shift $(l, m) \to (l + 1, m + 1)$ leaves p invariant and induces, by (2.1), the shift $n \to n + 1$, as one can see in figure 2. On the top-right square in figure 2 we can solve the equation,

$$Q(T^{p+1}u_{n+1}, T^{p+2}u_n, T^p u_{n+3}, T^{p+1}u_{n+2}; \alpha_{l+3}, \beta_{m+2}) = 0,$$

to find u_{n+3}, and hence the triple $(u_{n+1}, u_{n+2}, u_{n+3})$, from the triple (u_n, u_{n+1}, u_{n+2}) and the twist T. But this is not the only equation for u_{n+3}; considering the middle square in figure 2 we have

$$Q(T^p u_{n+1}, T^{p+1}u_n, T^{p-1}u_{n+3}, T^p u_{n+2}; \alpha_{l+1}, \beta_{m+1}) = 0. \tag{2.3}$$

which may also be used to find u_{n+3}. In general, if $\alpha_{l+2} = \alpha_l$ and $\beta_{m+1} = \beta_m$, then the reduction is consistent if T is chosen to be a symmetry of (1.3). In particular, if $\alpha_l = \alpha$ and $\beta_m = \beta$ are constants the resulting reductions are autonomous three-dimensional mappings.

To pass to the non-autonomous case, we note[6] that equations (1.6) and (1.8) only depend on the ratio α_l/β_m. For such *multiplicative* equations the reductions are consistent if $\alpha_{l+2}/\beta_{m+1} = \alpha_l/\beta_m$. Using separation of variables this yields

$$\frac{\alpha_{l+2}}{\alpha_l} = \frac{\beta_{m+1}}{\beta_m} := q^2, \tag{2.4}$$

which is a second order equation in α_l and first order in β_m. The general fully non-autonomous solution to (2.4) is

$$\alpha_l = \begin{cases} a_0 q^l & \text{if } l \text{ is even,} \\ a_1 q^l & \text{if } l \text{ is odd,} \end{cases} \qquad \beta_m = b_0 q^{2m}, \tag{2.5}$$

where we may absorb b_0 in a_0, a_1, or simply take $b_0 = 1$. The resulting reduction may be expressed in terms of $\beta_m/\alpha_l \propto q^n$.

Equation (1.7) may be written explicitly as a function of $\alpha_l - \beta_m$. For such *additive* equations, separation of variables yields

$$\alpha_{l+2} - \alpha_l = \beta_{m+1} - \beta_m := 2h. \tag{2.6}$$

The general fully non-autonomous solution to (2.6) is

$$\alpha_l = \begin{cases} a_0 + lh & \text{if } l \text{ is even,} \\ a_1 + lh & \text{if } l \text{ is odd,} \end{cases} \qquad \beta_m = b_0 + 2hm. \tag{2.7}$$

Here we may, without loss of generality, take $b_0 = 0$. In the additive case, the reduction will depend on $\alpha_l - \beta_m$ which depends linearly on the variable $n = 2m - l$.

For both these additive and multiplicative equations, the special reductions where a_i and b_i do not depend on i will be called simply non-autonomous. For the fully non-autonomous reductions the shift $n \to n+1$ has the effect of swapping the roles of a_0 and a_1. We have two options here: either to introduce a second root of unity or to consider the second iterate of the map. We choose the second option in this paper.

3. Twist matrices and Lax representations

In this section we will provide a method to construct Lax representations for twisted reductions. Firstly, let us consider a Lax pair for a lattice equation given by a pair of linear difference equations

$$\Psi_{l+1,m}(\gamma) = L_{l,m}(\gamma)\Psi_{l,m}(\gamma), \tag{3.1a}$$

$$\Psi_{l,m+1}(\gamma) = M_{l,m}(\gamma)\Psi_{l,m}(\gamma), \tag{3.1b}$$

where γ is a spectral parameter. This is a Lax pair in the sense that the compatibility condition between (3.1*a*) and (3.1*b*), which can be written as

$$L_{l,m+1}M_{l,m} - M_{l+1,m}L_{l,m} = 0, \tag{3.2}$$

is equivalent to imposing (1.3). For 3D-consistent equations of the form (1.3), see [3, 4], the matrices $L_{l,m}$ and $M_{l,m}$ are explicitly given in terms of derivatives of the function Q [37, equation (1.10)]. Therefore, and the importance of this will be apparent later on, because (1.6) and (1.8) are functions of α_l/β_m, the Lax matrices for (1.6) and (1.8), $L_{l,m}$ and $M_{l,m}$, will be functions of α_l/γ and β_m/γ, respectively. Similarly the Lax matrices $L_{l,m}$ and $M_{l,m}$ for (1.7) are functions of $\alpha_l - \gamma$ and $\beta_m - \gamma$, respectively.

To arrive at a particular form of the Lax pairs, we will sometimes perform a gauge transformation. For example, if the reduced equations can be dimensionally reduced by

[6] See table 1 in [37] for other equations of the ABS-list admitting such a representation.

choosing special variables, one would like to also express the Lax pair in terms of these variables. Then one considers

$$\Psi'_{l,m} = Z_{l,m}\Psi_{l,m}. \tag{3.3}$$

The linear system satisfied by $\Psi'_{l,m}$ is

$$\Psi'_{l+1,m} = (Z_{l+1,m}L_{l,m}Z_{l,m}^{-1})\Psi'_{l,m} = L'_{l,m}\Psi'_{l,m}, \tag{3.4a}$$

$$\Psi'_{l,m+1} = (Z_{l,m+1}M_{l,m}Z_{l,m}^{-1})\Psi'_{l,m} = M'_{l,m}\Psi'_{l,m}. \tag{3.4b}$$

In a slight abuse of notation, we will not distinguish between the pair $(L_{l,m}, M_{l,m})$ and $(L'_{l,m}, M'_{l,m})$.

Before we turn to the key ansatz we make in order to derive Lax pairs for the reduction, one must realize that if a solution to the lattice equation is known, $w_{l,m}$ for all $l, m \in \mathbb{Z}$, one can obtain a fundamental solution of the linear problem. Relating the behaviour of solutions of the nonlinear PDE with its spectral problem plays a fundamental role in inverse scattering methods for partial differential equations [2]. The discrete analogue of this theory for systems of difference equations has also been studied [7, 8] and applied to a system of the form (1.3) by Butler *et al* [6]. Our key anstatz is based on a relation between the solutions of systems defined by (1.3) and solutions of (3.1).

Let us start with the autonomous case. Given the fact that any solution, $w_{l,m}$, lifts to a solution of the linear problem, we may lift a solution satisfying (1.5) to a system that is now dependent on the variables u_n. That is to say, we have a solution to some linear system

$$\Psi_{l,m}(\gamma; \{w_{l,m}\}) \mapsto Y_n(\gamma; \{u_n\}).$$

We proceed as per usual, and construct operators, A_n and B_n, which are equivalent to shifts in l and m given by $(l, m) \to (l+2, m+1)$, and $(l, m) \to (l+1, m+1)$, respectively. These are given by the products

$$A_n(\gamma) \leftarrow\!\shortmid L_{l+1,m+1}L_{l,m+1}M_{l,m}, \tag{3.5a}$$

$$B_n(\gamma) \leftarrow\!\shortmid L_{l,m+1}M_{l,m}. \tag{3.5b}$$

The matrix A_n is called the monodromy matrix. It corresponds to a path, in figure 2, from u_n to Tu_n, going up one step and to the right two steps. We note that, in general, the matrix B_n is a particular factor of A_n, namely the one that corresponds to the shift $n \mapsto n+1$. The function $Y_n(\gamma; \{u_n\})$ now satisfies the equation

$$TY_n(\gamma; \{u_n\}) = A_n(\gamma)Y_n(\gamma; \{u_n\}) \tag{3.6a}$$

$$Y_{n+1}(\gamma; \{u_n\}) = B_n(\gamma)Y_n(\gamma; \{u_n\}), \tag{3.6b}$$

where, from the above, we may lift our symmetry, T, to the level of the linear problem via application on the $w_{l,m}$ (or equivalently on u_n).

Our *key ansatz* is that there is the additional relation

$$Y_n(\gamma; \{Tu_n\})Y_n(\gamma; \{u_n\})^{-1} = S_n(\{u_n\}), \tag{3.7}$$

where S_n does not depend on the spectral parameter. This rather innocuous looking relation implies that the singularities of Y_n, as a function of the spectral parameter γ, are independent of any particular solution of the lattice equation. That is, the singularities and poles of $Y_n(\gamma; \{Tu_n\})$ are cancelled out by the poles and singularities of $Y_n(\gamma; \{u_n\})^{-1}$ to give a constant matrix, $S_n(\{u_n\})$, which we call the *twist matrix*. For all examples of twisted reductions provided, we have been able to obtain such twist matrices.

Now, combining the two equations (3.6a) and (3.7) we obtain the first half of a standard Lax pair for an autonomous mapping

$$Y_n(\gamma) = S_n^{-1}A_n(\gamma)Y_n(\gamma). \tag{3.8}$$

where the other half of the Lax pair is (3.6*b*). The compatibility between (3.8) and (3.6*b*), which is equivalent to the autonomous reduction, is given by

$$S_{n+1}^{-1}A_{n+1}(\gamma)B_n(\gamma) - B_n(\gamma)S_n^{-1}A_n(\gamma) = 0, \tag{3.9}$$

and integrals for this reduction can be obtained by taking the trace of the twisted monodromy matrix $S_n^{-1}A_n(\gamma)$.

While A_n and B_n are determined by (3.5*a*) and (3.5*b*), the task of determining S_n remains. As is typical in integrable systems, the linear system is overdetermined, which gives us a straightforward, albeit complicated, way of calculating S_n. The complication arises because one needs to simultaneously calculate the twist matrix and the evolution equation from the compatibility condition, thereby increasing the number of conditions that need to be satisfied without increasing the number of relations from the compatibility. However, there is a simpler way to calculate S_n; observe that when we use (3.7), (3.6*a*) and (3.6*b*), we obtain

$$TY_{n+1} = T(B_n)A_nY_n = A_{n+1}B_nY_n.$$

Rewriting (3.9) yields the relation

$$A_{n+1}B_n = S_{n+1}B_nS_n^{-1}A_n.$$

By combining these equations, and by cancelling irrelevant factors, we obtain

$$T(B_n)S_n = S_{n+1}B_n, \tag{3.10}$$

which gives us an elegant way of calculating the twist matrix S_n and S_{n+1}, that does not rely explicitly on using the reduction. We will see that for the examples provided, the twist matrices are actually quite succinct. Furthermore, they tend to the identity matrix in the limit where the twists tends to the identity transformation.[7]

The non-autonomous case is a simple generalization of the above, since nothing we did relied upon any of the properties of α_l or β_m. We just need to specify a new spectral parameter for our reduced system. For the multiplicative equations, (1.6) and (1.8), we know that the $L_{l,m}$ and $M_{l,m}$ matrices are functions of α_l/γ and β_m/γ, respectively, which for our choices of parameters (2.5), can both be written in terms of q^l/γ and q^n only. This motivates the choice

$$x = q^l/\gamma, \tag{3.11}$$

as our spectral parameter. This implies that the shifts $(l, m) \to (l + 2, m + 1)$ and $(l, m) \to (l + 1, m + 1)$ both have the effect of translating x. As in the autonomous case, we may write $A_n(x)$ and $B_n(x)$ as products of matrices $L_{l,m}$ and $M_{l,m}$:

$$A_n(x) \leftarrow\!\!\dashv L_{l+1,m+1}L_{l,m+1}M_{l,m}, \tag{3.12a}$$

$$B_n(x) \leftarrow\!\!\dashv L_{l,m+1}M_{l,m}. \tag{3.12b}$$

where the linear problem, which is now in x, satisfies the equations

$$TY_n(q^2x) = A_n(x)Y_n(x), \tag{3.13a}$$

$$Y_{n+1}(qx) = B_n(x)Y_n(x), \tag{3.13b}$$

where, for the same reasons as above, we have the additional relation

$$TY_n(x) = S_nY_n(x). \tag{3.14}$$

This means our compatibility may be written

$$S_{n+1}^{-1}A_{n+1}(qx)B_n(x) - B_n(q^2x)S_n^{-1}A_n(x) = 0, \tag{3.15}$$

where S_n is actually the same twist matrix as in the autonomous case.

[7] Note that we also consider some examples of twists that are not homotopic to the identity transformation.

For the additive equation (1.7) the Lax matrices, $L_{l,m}$ and $M_{l,m}$, are functions of $\alpha_l - \gamma$ and $\beta_m - \gamma$, respectively. For the non-autonomous parameter choice, (2.7), $L_{l,m}$ and $M_{l,m}$ are both functions of $hl - \gamma$ and nh. This motivates the definition

$$x = hl - \gamma. \tag{3.16}$$

Using the same product formulae for $A_n(x)$ and $B_n(x)$, given by (3.12), the matrix $Y_n(x)$ satisfies the equations

$$TY_n(x+2h) = A_n(x)Y_n(x), \tag{3.17a}$$

$$Y_{n+1}(x+h) = B_n(x)Y_n(x), \tag{3.17b}$$

and also equation (3.14). This means that the compatibility yields

$$S_{n+1}^{-1}A_{n+1}(x+h)B_n(x) - B_n(x+2h)S_n^{-1}A_n(x) = 0. \tag{3.18}$$

In the following three sections we will provide the details for the (2,1)-reductions of our three examples to demonstrate this theory. We postpone the theory for general (s_1, s_2)-reduction to section 8.

4. Reductions of the lattice modified KdV equation

The discrete modified KdV equation (aka $H3_{\delta=0}$) , given by (1.6), was one of the earliest known integrable lattice equations. It appeared as a discrete analogue of the sine-Gordon equation (equivalent under a transformation) in the work of Hirota [24] and its Lax pair was derived using direct linearization [45]. Reductions of this equation have been considered by many authors [15, 17, 20, 29, 38, 44, 55, 57]. The equation has a Lax representation given by (3.1) where the Lax matrices are

$$L_{l,m}(\alpha_l/\gamma) = \begin{pmatrix} \dfrac{\gamma}{\alpha_l} & w_{l+1,m} \\ \dfrac{1}{w_{l,m}} & \dfrac{\gamma w_{l+1,m}}{w_{l,m}\alpha_l} \end{pmatrix}, \tag{4.1a}$$

$$M_{l,m}(\beta_m/\gamma) = \begin{pmatrix} \dfrac{\gamma}{\beta_m} & w_{l,m+1} \\ \dfrac{1}{w_{l,m}} & \dfrac{\gamma w_{l,m+1}}{w_{l,m}\beta_m} \end{pmatrix}. \tag{4.1b}$$

We will first recall how the autonomous periodic reductions are obtained, then proceed to generalize the reductions and their Lax representations to the twisted, and non-autonomous cases. In the periodic case, (1.5) is still valid, as is all the theory contained in sections 2 and 3, with the specialization to $T(w_{l,m}) = w_{l,m}$. This means that the labelling (2.2) is simply

$$w_{l,m} \mapsto u_{2m-l} = u_n.$$

In this case, the equation governing the reduction (2.3) is given by

$$u_{n+3} = \frac{u_n\,(\alpha u_{n+1} + \beta u_{n+2})}{\alpha u_{n+2} + \beta u_{n+1}}. \tag{4.2}$$

Using (3.5a) and (3.5b) we obtain the two Lax matrices, A_n and B_n, given by

$$A_n(\gamma) = \begin{pmatrix} \dfrac{\gamma}{\alpha} & u_n \\ \dfrac{1}{u_{n+1}} & \dfrac{\gamma u_n}{\alpha u_{n+1}} \end{pmatrix} B_n(\gamma),$$

$$B_n(\gamma) = \begin{pmatrix} \dfrac{\gamma}{\alpha} & u_{n+1} \\ \dfrac{1}{u_{n+2}} & \dfrac{\gamma u_{n+1}}{\alpha u_{n+2}} \end{pmatrix} \begin{pmatrix} \dfrac{\gamma}{\beta} & u_{n+2} \\ \dfrac{1}{u_n} & \dfrac{\gamma u_{n+2}}{\beta u_n} \end{pmatrix}.$$

We note that since $T(B_n) = B_n$, equation (3.10) becomes $B_n S_n = B_n S_{n+1}$, where S_n is *a priori* unknown. We parametrize S_n by letting

$$S_n = \begin{pmatrix} s_{1,n} & s_{2,n} \\ s_{3,n} & s_{4,n} \end{pmatrix}. \tag{4.3}$$

At the coefficient of γ^2, we obtain

$$s_{1,n+1} = s_{1,n}, \quad u_n s_{2,n} = u_{n+1} s_{2,n+1}, \quad u_{n+1} s_{3,n} = u_n s_{3,n+1}, \quad s_{4,n+1} = s_{4,n},$$

and at the coefficient of γ, we then obtain

$$s_{1,n} = s_{4,n}, \quad s_{2,n} = 0, \quad s_{3,n} = 0.$$

This tells us that we may choose $S_n = I$. Thus, the twisted monodromy matrix coincides with the standard monodromy matrix, which should not come as a surprise. Taking the trace of the monodromy matrix gives us $\alpha\beta\mathrm{Tr}(A_n) = \frac{2}{\alpha}\gamma^3 + K_{(4.2)}\gamma$ where

$$K_{(4.2)} = \alpha\left(\frac{u_n}{u_{n+2}} + \frac{u_{n+2}}{u_n}\right) + \beta\left(\frac{u_n}{u_{n+1}} + \frac{u_{n+1}}{u_n} + \frac{u_{n+1}}{u_{n+2}} + \frac{u_{n+2}}{u_{n+1}}\right)$$

is an integral, or constant of motion, of equation (4.2). One can verify that (3.9) is satisfied on solutions of (4.2). This equation, under the transformation $y_n = u_{n+1}/u_n$, takes the more familiar form of a second order difference equation

$$y_{n+1} y_n y_{n-1} = \frac{\alpha + \beta y_n}{\beta + \alpha y_n}, \tag{4.4}$$

which is more clearly a mapping of QRT type [46, 47]. The integral $K_{(4.2)}$ is also invariant under scaling and hence can be also written in terms of the reduced variable y_n,

$$K_{(4.4)} = \alpha\left(y_n y_{n+1} + \frac{1}{y_n y_{n+1}}\right) + \beta\left(y_n + \frac{1}{y_n} + y_{n+1} + \frac{1}{y_{n+1}}\right).$$

This reduction appeared in [44]. We will now give a one-parameter integrable generalization of this reduction by considering the twisted case.

The twist we apply is given by $T(w_{l,m}) = \lambda w_{l,m}$, which means that

$$w_{l,m} \mapsto \lambda^{l-m} u_{2m-l} = \lambda^p u_n.$$

Under this identification, the reduction (2.3) is given by

$$u_{n+3} = \frac{\lambda^2 u_n (\alpha u_{n+1} + \beta u_{n+2})}{\alpha u_{n+2} + \beta u_{n+1}}, \tag{4.5}$$

To obtain a Lax pair, we construct the operators A_n and B_n, using the product representation, (3.5*a*) and (3.5*b*), to give

$$A_n(\gamma) = \begin{pmatrix} \dfrac{\gamma}{\alpha} & \lambda u_n \\ \dfrac{1}{u_{n+1}} & \dfrac{\gamma\lambda u_n}{\alpha u_{n+1}} \end{pmatrix} B_n(\gamma),$$

$$B_n(\gamma) = \begin{pmatrix} \dfrac{\gamma}{\alpha} & u_{n+1} \\ \dfrac{\lambda}{u_{n+2}} & \dfrac{\gamma\lambda u_{n+1}}{\alpha u_{n+2}} \end{pmatrix} \begin{pmatrix} \dfrac{\gamma}{\beta} & \dfrac{u_{n+2}}{\lambda} \\ \dfrac{1}{u_n} & \dfrac{\gamma u_{n+2}}{\beta\lambda u_n} \end{pmatrix}.$$

We play the same game, where S_n is *a priori* unknown, hence, we let S_n be given by (4.3). The coefficient of γ^2 in (3.10) gives us the same conditions as in the periodic case, and at the coefficient of γ we find

$$s_{1,n} = \lambda s_{4,n}, \quad s_{2,n} = 0, \quad s_{3,n} = 0.$$

This gives us our first non-trivial twist matrix, given by

$$S_n = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.6}$$

Taking the trace of the twisted monodromy matrix gives us

$$\alpha\beta \mathrm{Tr}(S_n^{-1} A_n) = \left(\frac{1}{\lambda\alpha} + \frac{\lambda}{\alpha} \right) \gamma^3 + K_{(4.5)}\gamma,$$

where

$$K_{(4.5)} = \alpha \left(\frac{\lambda u_n}{u_{n+2}} + \frac{u_{n+2}}{\lambda u_n} \right) + \beta \left(\frac{\lambda u_n}{u_{n+1}} + \frac{u_{n+1}}{\lambda u_n} + \frac{\lambda u_{n+1}}{u_{n+2}} + \frac{u_{n+2}}{\lambda u_{n+1}} \right)$$

is an integral for (4.5). In the limit as $\lambda \to 1$, we retrieve the periodic case, making this a nice one-parameter family of reductions and their Lax pairs and integrals. This provides all the elements for (3.9) to give (4.5). Once again, by identifying $y_n = u_{n+2}/u_{n+1}$, we have the classic QRT map

$$y_{n+1} y_n y_{n-1} = \frac{\lambda^2(\alpha + \beta y_n)}{\beta + \alpha y_n}, \tag{4.7}$$

with corresponding integral obtained from $K_{(4.5)}$. Thus, we have obtained a one-parameter generalization of the reduction (4.2) found in [44].

When we turn to the simply non-autonomous case, we obtain a version of q-P_{II}. In taking $\alpha_l = aq^l$ and $\beta_m = q^{2m}$, we need to take into account the position of the square we use to evaluate the reduction. With respect to figure (2.2), if the square whose lower left entry is u_n denotes (l, m), the relevant square used for (2.3) is at $(l+1, m+1)$ Thus, we obtain the reduction

$$u_{n+3} = \frac{\lambda^2 u_n \left(a u_{n+1} + q^{n+1} u_{n+2} \right)}{a u_{n+2} + q^{n+1} u_{n+1}}. \tag{4.8}$$

We now use (3.11) in our product representation for $A_n(x)$ and $B_n(x)$, to obtain

$$A_n(x) = \begin{pmatrix} \frac{1}{qxa} & \lambda u_n \\ \frac{1}{u_{n+1}} & \frac{\lambda u_n}{qxau_{n+1}} \end{pmatrix} B_n(x),$$

$$B_n(x) = \begin{pmatrix} \frac{1}{xa} & u_{n+1} \\ \frac{\lambda}{u_{n+2}} & \frac{\lambda u_{n+1}}{xau_{n+2}} \end{pmatrix} \begin{pmatrix} \frac{1}{xq^n} & \frac{u_{n+2}}{\lambda} \\ \frac{1}{u_n} & \frac{u_{n+2}}{x\lambda q^n u_n} \end{pmatrix}.$$

We use the form (4.3) once more, and the calculations follow analogously to the previous case and give (4.6). With $A_n(x)$, $B_n(x)$ and S_n defined, the compatibility, (3.15), gives (4.8). Furthermore, by letting $y_n = u_{n+2}/u_{n+1}$, we find a more direct correspondence with a q-analogue of the second Painlevé equation found in [50],

$$y_{n+1} y_n y_{n-1} = \frac{\lambda^2(a + q^{n+1} y_n)}{q^{n+1} + a y_n}, \tag{4.9}$$

which generalizes a reduction of Nijhoff and Papageorgiou [29]. At this point, we note that we may use alternative Lax matrices to (4.1). By considering a transformation of the form (3.3), where

$$Z_{l,m} = \begin{pmatrix} \dfrac{1}{w_{l,m}} & 0 \\ 0 & 1 \end{pmatrix}$$

we obtain a Lax pair given by

$$L_{l,m} = \begin{pmatrix} \dfrac{\gamma w_{l,m}}{\alpha_l w_{l+1,m}} & 1 \\ 1 & \dfrac{\gamma w_{l+1,m}}{\alpha_l w_{l,m}} \end{pmatrix}, \tag{4.10}$$

$$M_{l,m} = \begin{pmatrix} \dfrac{\gamma w_{l,m}}{\beta_m w_{l,m+1}} & 1 \\ 1 & \dfrac{\gamma w_{l,m+1}}{\beta_m w_{l,m}} \end{pmatrix}. \tag{4.11}$$

Note that these matrices are actually invariant under the uniform application of the transformation $w_{l,m} \to T(w_{l,m})$. Since all the variables $w_{l,m}$ in $L_{l,m}$ and $M_{l,m}$ appear in ratios, the Lax pair may be expressed in the variables $y_n = u_{n+2}/u_{n+1}$. In this light, we write an alternative set of Lax matrices

$$A_n(x) = \begin{pmatrix} \dfrac{y_{n-1}}{qx\lambda a} & 1 \\ 1 & \dfrac{\lambda}{qxay_{n-1}} \end{pmatrix} B_n(x),$$

$$B_n(x) = \begin{pmatrix} \dfrac{y_n}{x\lambda a} & 1 \\ 1 & \dfrac{\lambda}{xay_n} \end{pmatrix} \begin{pmatrix} \dfrac{\lambda}{q^n x y_n y_{n-1}} & 1 \\ 1 & \dfrac{y_n y_{n-1}}{q^n x \lambda} \end{pmatrix},$$

and twist matrix $S_n = I$. This is an immediate consequence of the fact that y_n is an invariant of T: $T(y_n) = T(u_{n+2})/T(u_{n+1}) = u_{n+2}/u_{n+1} = y_n$. We remark that twist matrices are not gauge invariant.

The last case to do is the fully non-autonomous generalization, where α_l and β_m are given by (2.5) with $b_0 = 1$. It should be noted that the resulting equation governing $n \to n+1$ turns an even l into an odd l, hence, the evolution equation incorporates a change in a_0 and a_1. With this in mind, the evolution equation is given by (2.3) combined with a change in a_0 and a_1: in the case that n (and hence, l) is even, u_{n+3} is calculated from

$$u_{n+3} = \frac{\lambda^2 u_n \left(a_1 u_{n+1} + q^{n+2} u_{n+2}\right)}{a_1 u_{n+2} + q^{n+2} u_{n+1}}, \quad a_0 \to \frac{a_1}{q}, \quad a_1 \to q a_0. \tag{4.12}$$

This system possesses a Lax pair of the form (3.13), where the Lax matrices are given by products (3.12*a*) and (3.12*b*). The shift $n \to n+2$ has an alternative deformation matrix, given by $B_n(x) \leftarrow\!\!\dashv M_{l,m}$, which simplifies the calculation. If we let $A_n(x)$ be given by the product (3.12*a*), we obtain

$$A_n(x) = \begin{pmatrix} \dfrac{1}{xa_1} & \lambda u_n \\ \dfrac{1}{u_{n+1}} & \dfrac{\lambda u_n}{xa_1 u_{n+1}} \end{pmatrix} \begin{pmatrix} \dfrac{1}{xa_0} & u_{n+1} \\ \dfrac{\lambda}{u_{n+2}} & \dfrac{\lambda u_{n+1}}{xa_0 u_{n+2}} \end{pmatrix} B_n(x),$$

$$B_n(x) = \begin{pmatrix} \frac{1}{xq^n} & \frac{u_{n+2}}{\lambda} \\ \frac{1}{u_n} & \frac{u_{n+2}}{x\lambda q^n u_n} \end{pmatrix}.$$

Using (4.3) and (3.10) we once again obtain (4.6). Here the compatibility condition is

$$S_{n+1}^{-1}A_{n+1}(x)B_n(x) - B_n(q^2x)S_n^{-1}A_n(x) = 0, \tag{4.13}$$

which we use to obtain (4.12). However, this is not as obviously a two-dimensional mapping. We employ a technique used in [37] to rewrite this equation. We take $A_n(x)$ and evaluate the root of the upper right entry (in x^2), denoting this y. The determinant of $A_n(\sqrt{y})$ factors nicely, and the factors of det $A_n(\sqrt{y})$ appear in the diagonal entries, in addition to a simple multiplicative factor, which we denote z_n. Explicitly, modulo some scaling, these variables are

$$y_n = \frac{a_1 u_{n+1}}{u_n} + \frac{a_0 u_{n+1}u_{n+2} + q^n u_n u_{n+2}}{\lambda^2 u_n^2}, \tag{4.14a}$$

$$z_n = u_{n+1}\left(\frac{a_1\lambda u_{n+1}}{u_{n+2}} + \frac{a_0}{\lambda u_n}\right), \tag{4.14b}$$

which then satisfy the difference equations

$$y_n y_{n+2} = \left(\lambda q^{n+2} + z_n\right)\left(\frac{q^n}{\lambda} + z_n\right) \tag{4.15a}$$

$$z_n z_{n+2} = \frac{\left(a_1 q^{n+2} + a_0 y_{n+2}\right)\left(a_0 q^{n+2} + a_1 y_{n+2}\right)}{\left(a_0 a_1 + q^{n+2} y_{n+2}\right)}. \tag{4.15b}$$

This equation first appeared in the work of Ramani *et al* [52] and is related, via a Miura transformation, to a version of q-P$_{III}$ found in [26]. This equation has a symmetry group which is of affine Weyl type $A_2^{(1)} + A_1^{(1)}$ [14, 53].

Another possible choice of twist is $T_2 : w \to \lambda/w$, which is is not homotopic to the identity. The twist matrix associated with (s_1, s_2)-reductions of (1.6) with fixed Lax representation (4.1) is

$$S_n = \begin{pmatrix} 0 & \lambda \\ 1 & 0 \end{pmatrix},$$

for a large class of s_1 and s_2.

5. Reductions of the lattice potential KdV equation

The lattice potential KdV equation (aka $H1$) (1.7) was derived from the direct linearization approach [30], and it yields the potential KdV equation in a continuum limit. Periodic reductions of (1.7) were considered by many authors [20, 29, 32, 39, 44, 58]. The (2, 1)-periodic non-autonomous reduction and its Lax pair were recently given in [37].

The equation (1.7) has a Lax representation given by (3.1) where the Lax matrices are given by

$$L_{l,m} = \begin{pmatrix} w_{l,m} & -\gamma + \alpha_l - w_{l,m}w_{l+1,m} \\ 1 & -w_{l+1,m} \end{pmatrix}, \tag{5.1a}$$

$$M_{l,m} = \begin{pmatrix} w_{l,m} & -\gamma + \beta_m - w_{l,m}w_{l,m+1} \\ 1 & -w_{l,m+1} \end{pmatrix}. \tag{5.1b}$$

The twist that we seek to apply is the transformation $T(w_{l,m}) = w_{l,m} + \lambda$, which means our reduced variables are specified by

$$w_{l,m} \mapsto u_{2m-l} + (l-m)\lambda = u_n + p\lambda.$$

For the twisted autonomous case, where $\alpha_l = \alpha$ and $\beta_m = \beta$ are constants, it is clear that we obtain the difference equation

$$(u_n - u_{n+3} + 2\lambda)(u_{n+1} - u_{n+2}) = \alpha - \beta. \tag{5.2}$$

The Lax pair for this autonomous equation may be specified by (3.5*a*) and (3.5*b*), where the lattice variables take on their reduced values, giving

$$A_n(\gamma) = \begin{pmatrix} u_{n+1} & \alpha - \gamma - (\lambda + u_n)u_{n+1} \\ 1 & -(\lambda + u_n) \end{pmatrix} B_n(\gamma),$$

$$B_n(\gamma) = \begin{pmatrix} -\lambda + u_{n+2} & \alpha - \gamma - (u_{n+2} - \lambda)u_{n+1} \\ 1 & -u_{n+1} \end{pmatrix} \begin{pmatrix} u_n & \beta - \gamma - (u_{n+2} - \lambda)u_n \\ 1 & \lambda - u_{n+2} \end{pmatrix}.$$

We now need to calculate S_n, which is once again, *a priori*, an unknown function of n, hence, we label the elements of S_n by (4.3). By utilizing (3.10), at the level of the coefficient of γ, we find $S_{n+1} = S_n$. Solving for the constant coefficient of (3.10) gives us that if S_n is independent of n, then $s_{3,n} = 0$ and $s_{2,n} = \lambda s_{1,n}$ and $s_{4,n} = s_{1,n}$, which we may simplify to give the second non-trivial twist matrix in this study, given by

$$S_n = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}. \tag{5.3}$$

Knowing $A_n(\gamma)$, $B_n(\gamma)$ and S_n gives us all the necessary ingredients for calculating the compatibility (3.9), which gives us the required mapping (5.2). Calculating the trace of the twisted monodromy matrix, $\mathrm{Tr}(S_n^{-1}A_n) = 2\lambda\gamma + K_{(5.2)}$, we obtain an integral,

$$K_{(5.2)} = \alpha(u_n - u_{n+2}) + \beta(u_{n+2} - u_n - 2\lambda) + (u_{n+1} - u_{n+2})(u_n - u_{n+1})(u_{n+2} - u_n - 2\lambda). \tag{5.4}$$

Note that once again S_n has the property that as $\lambda \to 0$, $S_n \to I$, giving the periodic case.

To simply de-autonomize the lattice equation and the Lax pair, we let $\alpha_l = a + lh$ and $\beta_m = 2mh$, in which case the reduction (2.3) becomes

$$u_{n+3} - u_n = \frac{a - hn - h}{u_{n+2} - u_{n+1}} + 2\lambda, \tag{5.5}$$

which we may transform to be a function of $y_n = u_{n+2} - u_{n+1}$, giving

$$y_{n+1} + y_n + y_{n-1} = \frac{a - hn - h}{y_n} + 2\lambda. \tag{5.6}$$

This is a form of d-P$_I$ (see [51]) and generalizes the reduction found in [37]. Furthermore, the method we present also gives us the Lax pair for this reduction. We specify our spectral parameter, given by (3.16), and construct $A_n(x)$ and $B_n(x)$ via their product representations, (3.12*a*) and (3.12*b*), to give

$$A_n(x) = \begin{pmatrix} u_{n+1} & h + x + a - u_{n+1}(u_n + \lambda) \\ 1 & -u_n - \lambda \end{pmatrix} B_n(x),$$

$$B_n(x) = \begin{pmatrix} u_{n+2} - \lambda & x + a - u_{n+1}(u_{n+2} - \lambda) \\ 1 & \lambda - u_{n+2} \end{pmatrix} \begin{pmatrix} u_n & hn + x - u_n(u_{n+2} - \lambda) \\ 1 & \lambda - u_{n+2} \end{pmatrix}.$$

Once again, we assume that S_n is unknown, hence, we let S_n be given by (4.3). Then, using (3.10), we find that S_n is given by (5.3). This gives us all the required elements of (3.18), which in turn, gives us (5.5).

As in the modified KdV reduction, it is possible to apply a transformation of the form of (3.3), where

$$Z_{l,m} = \begin{pmatrix} 1 & w_{l,m} \\ 0 & 1 \end{pmatrix}$$

to give the alternative Lax matrices,

$$\begin{aligned} L_{l,m} &= \begin{pmatrix} w_{l,m} - w_{l+1,m} & (w_{l,m} - w_{l+1,m})^2 + \alpha_l - \gamma \\ 1 & w_{l,m} - w_{l+1,m} \end{pmatrix}, \\ M_{l,m} &= \begin{pmatrix} w_{l,m} - w_{l,m+1} & (w_{l,m} - w_{l,m+1})^2 + \beta_m - \gamma \\ 1 & w_{l,m} - w_{l,m+1} \end{pmatrix}, \end{aligned}$$

which have the desirable property of being expressed in terms of differences of the variables $w_{l,m}$. This means, these matrices admit a parametrization in terms of Painlevé variables, $y_n = u_{n+2} - u_{n+1}$,

$$A_n(x) = \begin{pmatrix} y_{n-1} - \lambda & a + x + h + (\lambda - y_{n-1})^2 \\ 1 & y_{n-1} - \lambda \end{pmatrix} B_n(x),$$

$$B_n(x) = \begin{pmatrix} y_n - \lambda & a + x + (\lambda - y_n)^2 \\ 1 & y_n - \lambda \end{pmatrix} \begin{pmatrix} \lambda - y_{n-1} - y_n & x + nh + (\lambda - y_{n-1} - y_n)^2 \\ 1 & \lambda - y_n - y_{n-1} \end{pmatrix}.$$

We note that the transformation, T, applied to y_n is trivial, just as in the previous section. This gives us that $S_n = I$, and the compatibility (3.15) gives us (5.6). This is not the first Lax pair known for equation (5.6), as a 3×3 Lax pair was derived in the work of Papageorgiou *et al* [40]. We do not know whether a 2×2 Lax pair, such as the one presented, is known or not.

We now turn to the fully non-autonomous twisted periodic reduction, where the α_l and β_m variables are given by (2.7), with $b_0 = 0$. It was recently noted that the fully non-autonomous periodic reduction may be identified as a special case of the discrete analogue of the fourth Painlevé equation [37]. We expect this to be the case again.

As before, the evolution equations must be taken into account the way in which the $n \to n+1$ shift changes l from an even number to an odd number, because the roles of a_0 and a_1 change every single iteration. The evolution equation (2.3) in this case is given by

$$u_{n+3} - u_n = \frac{-a_1 + hn + h}{u_{n+1} - u_{n+2}} + 2\lambda, \qquad a_0 \to a_1 - h, \qquad a_1 \to a_0 + h. \quad (5.7)$$

Once again, it is not obvious that the mapping associated with the shift $n \to n+2$ is a two-dimensional mapping. But we can find reduced variables y_n and z_n, by exploiting the Lax matrices for the equation, which are

$$A_n(x) = \begin{pmatrix} u_{n+1} & x + a_1 - u_{n+1}(u_n + \lambda) \\ 1 & -u_n - \lambda \end{pmatrix} B_n(x),$$

$$B_n(x) = \begin{pmatrix} u_{n+2} - \lambda & x + a_0 - u_{n+1}(u_{n+2} - \lambda) \\ 1 & -u_{n+1} \end{pmatrix} \begin{pmatrix} u_n & x + nh - u_n(u_{n+2} - \lambda) \\ 1 & \lambda - u_{n+2} \end{pmatrix}.$$

The variables are explicitly given by

$$\begin{aligned} y_n &= -a_0 + (u_n - u_{n+1})(2\lambda + u_n - u_{n+2}), \\ z_n &= \frac{a_0 + y_n}{u_n - u_{n+1}}. \end{aligned}$$

These two functions of the lattice variables satisfy

$$y_{n+2} + y_n = z_n(z_n - 2\lambda) - a_0 - a_1, \quad (5.8a)$$

$$z_{n+2} z_n = -\frac{(y_{n+2} + a_0)(y_{n+2} + a_1)}{y_{n+2} + h(n+2)}, \quad (5.8b)$$

which is a discrete version of the fourth Painlevé equation found in [50, 52]. This is a one-parameter family of reductions that generalizes the one presented in [37].

On the other hand, equation (5.7) is equivalent to asymmetric d-$\mathrm{P_I}$, see [52, equation (3.33)], where a relation to d-P_{IV} was obtained through a quadratic transformation. In fact, taking

$$\alpha_l = a_1 + a_2(-1)^l + hl,$$

instead of (2.7), the equation then becomes

$$u_{n+3} - u_n = \frac{h(n+1) - a_1 + a_2(-1)^n}{u_{n+1} - u_{n+2}} + 2\lambda,$$

or, if we let $y_n = u_{n+1} - u_n$, this becomes

$$y_{n+1} + y_n + y_{n-1} = \frac{hn - a_1 - a_2(-1)^n}{y_n} + 2\lambda,$$

which is the most general form of d-P_I [51]. In the autonomous limit, taking $h = 0$, which would correspond to the 'fully autonomous case', the equation admits the following integral:

$$K_{(5.2)} + a_2(-1)^n(2u_{n+1} - u_n - u_{n+2}),$$

where $K_{(5.2)}$ is given in (5.4), taking $\alpha = a_1$ and $\beta = 0$.

Just as we did for (1.6), we present a twist matrix for a twist that is not homotopic to the identity twist, namely the twist $T_2 : w \to \lambda - w$. The twist matrix associated with (s_1, s_2)-reductions of (1.7) with a fixed Lax representation (5.1) is

$$S_n = \begin{pmatrix} -1 & -\lambda \\ 0 & 1 \end{pmatrix},$$

for a number of different choices of s_1 and s_2. This twist also yields a class of integrable mappings and their Lax representations.

6. Reductions of the lattice Schwarzian KdV equation

Periodic reductions of the lattice Schwarzian KdV equation (aka $Q1_{\delta=0}$), given by (1.8), have been the subject of a number of studies [19, 32, 55]. Most recently, three of the authors considered periodic reductions that gave rise to q-P_{VI} and q-$\mathrm{P}(A_2^{(1)})$ [37].

A Lax pair for equation (1.8) is of the form (3.1) where the Lax matrices are

$$L_{l,m} = \begin{pmatrix} 1 & w_{l,m} - w_{l+1,m} \\ \dfrac{\alpha}{\gamma(w_{l,m} - w_{l+1,m})} & 1 \end{pmatrix},$$

$$M_{l,m} = \begin{pmatrix} 1 & w_{l,m} - w_{l,m+1} \\ \dfrac{\beta}{\gamma(w_{l,m} - w_{l,m+1})} & 1 \end{pmatrix}.$$

From our perspective, (1.8) is of particular interest, as it is invariant under the full group of Möbius transformations, denoted $\mathrm{PGL}(2, \mathbb{C})$. We parametrize each Möbius transformation in terms of its fixed points, τ_1 and τ_2, and the eigenvalues, λ_1 and λ_2, of a corresponding matrix, as follows:

$$T(w) = \frac{(\lambda_1\tau_1 - \lambda_2\tau_2)w - (\lambda_1 - \lambda_2)\tau_1\tau_2}{(\lambda_1 - \lambda_2)w + \lambda_2\tau_1 - \lambda_1\tau_2}.$$

The reduced variables are given nicely in terms of τ_1, τ_2, λ_1 and λ_2 as

$$w_{l,m} \mapsto T^{l-m}u_{2m-l} = T^p u_n = \frac{(\lambda_1^p\tau_1 - \lambda_2^p\tau_2)u_n - (\lambda_1^p - \lambda_2^p)\tau_1\tau_2}{(\lambda_1^p - \lambda_2^p)u_n + \lambda_2^p\tau_1 - \lambda_1^p\tau_2}.$$

It will often be more notationally convenient to use the symbolic notation $T^p u_n$ over the explicit expression for obvious reasons. In the autonomous case, where $\alpha_l = \alpha$ and $\beta_m = \beta$ are constants, the reduced equation may be expressed as

$$u_{n+3} = \frac{\alpha T^2u_n Tu_{n+1} - Tu_{n+2}((\alpha-\beta)Tu_{n+1} + \beta T^2u_n)}{(\alpha-\beta)T^2u_n - \alpha Tu_{n+2} + \beta Tu_{n+1}}. \tag{6.1}$$

We form the Lax pair in the usual manner, where (3.5*a*) and (3.5*b*) give us the following representations for $A_n(\gamma)$ and $B_n(\gamma)$:

$$A_n(\gamma) = \begin{pmatrix} 1 & u_{n+1} - Tu_n \\ \dfrac{\alpha}{\gamma(u_{n+1} - Tu_n)} & 1 \end{pmatrix} B_n,$$

$$B_n(\gamma) = \begin{pmatrix} 1 & T^{-1}u_{n+2} - u_{n+1} \\ \dfrac{\alpha}{\gamma(T^{-1}u_{n+2} - u_{n+1})} & 1 \end{pmatrix} \begin{pmatrix} 1 & u_n - T^{-1}u_{n+2} \\ \dfrac{\beta}{\gamma(u_n - T^{-1}u_{n+2})} & 1 \end{pmatrix}.$$

The calculation of the twist matrix is algebraically more difficult than in the previous cases, but essentially follows the same logic. That is, we let S_n be given by (4.3) and use (3.10) at the various coefficients. The calculations are much simpler if one assumes (6.1), but it is not necessary to do so. It is also useful to compare the iterates of the entries of S_n with the calculated values for S_{n+1}. This gives us our third non-trivial twist matrix, associated with the Möbius transformation, given by

$$S_n = \begin{pmatrix} \dfrac{\lambda_1\lambda_2(\tau_1 - \tau_2)}{\lambda_1(u_n - \tau_2) - \lambda_2(u_n - \tau_1)} & 0 \\ \dfrac{\lambda_1 - \lambda_2}{\tau_1 - \tau_2} & \dfrac{\lambda_1(u_n - \tau_2) - \lambda_2(u_n - \tau_1)}{\tau_1 - \tau_2} \end{pmatrix}. \tag{6.2}$$

The coefficient of γ^{-1} in the trace of the twisted monodromy matrix provides the following integral for equation (6.1):

$$\begin{aligned} K_{(6.1)} = {} & \frac{\alpha(Tu_{n+1} - T^2u_n)(\lambda_1(\tau_2 - u_{n+2}) - \lambda_2(\tau_1 - u_{n+2}))}{\lambda_1\lambda_2(\tau_1 - \tau_2)(Tu_{n+1} - u_{n+2})} \\ & + \frac{\beta(u_{n+2} - T^2u_n)(\lambda_1(Tu_n - \tau_2) - \lambda_2(Tu_n - \tau_1))}{\lambda_1\lambda_2(\tau_1 - \tau_2)(Tu_n - u_{n+2})} \\ & + \frac{\alpha(\tau_1 - \tau_2)(Tu_nT^2u_n - (Tu_{n+1})^2 + u_{n+2}(2Tu_{n+1} - Tu_n - T^2u_n))}{(\lambda_1(Tu_n - \tau_2) - \lambda_2(Tu_n - \tau_1))(Tu_{n+1} - T^2u_n)(Tu_{n+1} - u_{n+2})}. \end{aligned}$$

We have determined the reduced variables to be

$$y_n = \frac{\beta\,(Tu_n - u_n)\left(T^{-1}u_{n+2} - u_{n+1}\right)}{\alpha\,(Tu_n - u_{n+1})\left(T^{-1}u_{n+2} - u_n\right)},$$

$$z_n = \frac{\lambda_1\lambda_2\,(\tau_1 - \tau_2)\,(\alpha y_n - 1)(T^{-1}u_{n+2} - u_n)}{(Tu_n - T^{-1}u_{n+2})\,(\lambda_2\,(\tau_1 - u_n) + \lambda_1\,(u_n - \tau_2))},$$

and hence we obtain the equation

$$y_{n+1}y_n = \frac{\beta\,(z_n - \lambda_1)\,(z_n - \lambda_2)}{\alpha\lambda_1\lambda_2}, \tag{6.3a}$$

$$z_{n+1}z_n = (1 - y_{n+1})\lambda_1\lambda_2, \tag{6.3b}$$

which is of QRT type and admits the integral

$$K_{6.3} = \alpha\left(\frac{y_n - 1}{z_n} - \frac{z_n}{\lambda_1\lambda_2}\right) + \beta\left(\frac{(z_n-\lambda_1)(z_n-\lambda_2)}{\lambda_1\lambda_2 y_n z_n} - \frac{1}{z_n}\right).$$

Let us jump right to the fully non-autonomous reduction, where the variables α_l and β_m are given by (2.5), with $b_0 = 1$. If we assume l (and hence n) is even, then the evolution equation is given by

$$a_1 \to q a_0, \quad a_0 \to \frac{a_1}{q}, \tag{6.4}$$

$$u_{n+3} = \frac{a_1 T u_{n+1}\left(Tu_{n+2} - T^2 u_n\right) + q^{n+2} T u_{n+2}\left(T^2 u_n - T u_{n+1}\right)}{a_1\left(T u_{n+2} - T^2 u_n\right) + q^{n+2}\left(T^2 u_n - T u_{n+1}\right)}.$$

The Lax matrices are given by (3.12*a*) and (3.12*b*),

$$A_n(x) = \begin{pmatrix} 1 & u_{n+1} - Tu_n \\ \dfrac{x a_1}{u_{n+1} - T u_n} & 1 \end{pmatrix} B_n(x),$$

$$B_n(x) = \begin{pmatrix} 1 & T^{-1}u_{n+2} - u_{n+1} \\ \dfrac{x a_0}{T^{-1}u_{n+2} - u_{n+1}} & 1 \end{pmatrix} \begin{pmatrix} 1 & u_n - T^{-1}u_{n+2} \\ \dfrac{x q^n}{u_n - T^{-1}u_{n+2}} & 1 \end{pmatrix}.$$

We use (3.10) to deduce that S_n is again given by (6.2). Using the compatibility, (3.15), we readily find (6.4). Once again, the task remains to find a second order system from this equation. We choose a similar combination of lattice variables as before, by letting

$$y_n = \frac{(Tu_n - u_n)(T^{-1}u_{n+2} - u_{n+1})}{a_0(Tu_n - u_{n+1})(T^{-1}u_{n+2} - u_n)},$$

$$z_n = \frac{(T^{-1}u_{n+2} - Tu_n)\left(\lambda_2(\tau_1 - u_n) + \lambda_1(u_n - \tau_2)\right)}{(\tau_1 - \tau_2)(T^{-1}u_{n+2} - u_n)}.$$

Under this change of variables, we obtain another version of the system obtained in [52], which generalizes (4.15),

$$y_{n+2} y_n = \frac{(z_n - \lambda_1)(z_n - \lambda_2)}{\lambda_1\lambda_2 a_0 a_1}, \tag{6.5a}$$

$$z_{n+2} z_n = \frac{\lambda_1\lambda_2 (a_0 y_{n+2} - 1)(a_1 y_{n+2} - 1)}{1 - q^{n+2} y_{n+2}}, \tag{6.5b}$$

modulo a certain scaling of variables. It is interesting to note that as a system admitting singularity confinement, the critical values of z_n depend explicitly on the eigenvalues of the twist.

7. (2, 2)-reduction, and q-P$_{VI}$

Three of the authors have presented two versions of q-P$_{VI}$, from (1.6) in [38] and from (1.8) in [37]. Both of these reductions were subcases of the system described in the work of Jimbo and Sakai [25]; the version in [37] appeared with an interesting biquadratic constraint, which was similar to the work of Yamada [59] but not present in [25], while the version in [38] is a subcase of the version in [37]. Here we will present the fully non-autonomous (2, 2)-reduction of (1.8), which we identify with the full parameter unconstrained version of the q-analogue of the sixth Painlevé equation as it appears in [25].

We start by specifying new n and p variables, which we assign to be

$$n = m - l, \quad p = \left\lfloor \frac{l}{2} \right\rfloor,$$

where $\lfloor x \rfloor$ rounds x down to the nearest integer. In this way, we label the variables $w_{l,m}$ so that

$$w_{l,m} \mapsto \begin{cases} T^p u_n & \text{if } l \text{ is even,} \\ T^p v_n & \text{if } l \text{ is odd.} \end{cases} \tag{7.1}$$

This labelling is depicted in figure 3.

In order for this system to be consistent, we require

$$\frac{\alpha_{l+2}}{\beta_{m+2}} = \frac{\alpha_l}{\beta_m},$$

which we solve by letting

$$\alpha_l = \begin{cases} a_0 q^l & \text{if } l \text{ is even,} \\ a_1 q^l & \text{if } l \text{ is odd,} \end{cases} \quad \beta_m = \begin{cases} b_0 q^m & \text{if } m \text{ is even,} \\ b_1 q^m & \text{if } m \text{ is odd.} \end{cases}$$

We now pick a spectral variable, $x = q^l/\gamma$, in which we have the system of linear equations

$$\begin{aligned} T Y_n(q^2 x) &= A_n(x) Y_n(x), \\ Y_n(x) &= B_n(x) Y_n(x), \end{aligned}$$

where the spectral matrix, $A_n(x)$, governs an operation that is equivalent to the shift $(l, m) \to (l+2, m+2)$ and the deformation matrix, $B_n(x)$, governs an operation that is equivalent to the shift $(l, m) \to (l, m+1)$. This gives us a linear system with Lax matrices

$$\begin{aligned} A_n(x) &\leftarrow\!\dashv L_{l+1,m+2} M_{l+1,m+1} L_{l,m+1} M_{l,m}, \\ B_n(x) &\leftarrow\!\dashv M_{l,m}. \end{aligned}$$

explicitly given by

$$\begin{aligned} A_n(x) = &\begin{pmatrix} 1 & v_{n+1} - T u_n \\ \dfrac{x a_1}{v_{n+1} - T u_n} & 1 \end{pmatrix} \begin{pmatrix} 1 & v_n - v_{n+1} \\ \dfrac{q^n x b_1}{v_n - v_{n+1}} & 1 \end{pmatrix} \\ &\times \begin{pmatrix} 1 & u_{n+1} - v_n \\ \dfrac{x a_0}{u_{n+1} - v_n} & 1 \end{pmatrix} B_n(x), \\ B_n(x) = &\begin{pmatrix} 1 & u_n - u_{n+1} \\ \dfrac{q^n x b_0}{u_n - u_{n+1}} & 1 \end{pmatrix}. \end{aligned}$$

The twist matrix S_n is the same as in the (2,1)-reduction, given by (6.2), and the compatibility condition

$$S_{n+1}^{-1} A_{n+1}(x) B_n(x) - B_n(q^2 x) S_n^{-1} A_n(x) = 0$$

gives the system that fixes the a_0 and a_1, and induces the transformation

$$\begin{aligned} &b_0 \to \frac{b_1}{q}, \quad b_1 \to q b_0, \\ u_{n+2} &= \frac{a_0 u_{n+1} (v_n - v_{n+1}) + b_1 q^n v_{n+1} (u_{n+1} - v_n)}{a_0 (v_n - v_{n+1}) + b_1 q^n (u_{n+1} - v_n)}, \\ v_{n+2} &= \frac{a_1 v_{n+1} (T u_n - T u_{n+1}) + b_0 q^{n+2} T u_{n+1} (v_{n+1} - T u_n)}{a_1 (T u_n - T u_{n+1}) + b_0 q^{n+2} (v_{n+1} - T u_n)}. \end{aligned}$$

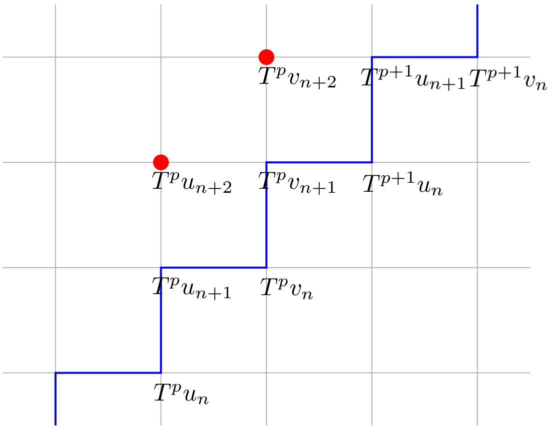

Figure 3. Labelling of variables for the (2,2)-reduction of the lattice with respect to (7.1).

This system possesses a 2-integral[8], which we label κ, given by

$$\kappa = \frac{b_0 q^n (u_{n+1} - v_n)(\lambda_2(\tau_1 - u_n) + \lambda_1(u_n - \tau_2))(Tu_n - v_{n+1})}{a_1\lambda_1(\tau_2 - \tau_1)(u_n - u_{n+1})(v_n - v_{n+1})}.$$

The $A_n(x)$ may be identified with the parametrization of the spectral matrix of Jimbo and Sakai (see [25]) with variables y_n and z_n specified by

$$y_n = \frac{b_0 q^n (v_n - u_{n+1})(\lambda_2(\tau_1 - u_n) + \lambda_1(u_n - \tau_2))(Tu_n - v_{n+1})}{a_1\lambda_1(\tau_1 - \tau_2)(u_n - u_{n+1})(v_n - v_{n+1})},$$

$$z_n = \frac{(v_{n+1} - v_n)(Tu_n - u_n)(u_{n+1} - v_n)}{a_0(u_n - u_{n+1})(v_n - v_{n+1})(Tu_n - v_n) + b_1 q^n (u_n - v_n)(u_{n+1} - v_n)(Tu_n - v_{n+1})}.$$

Under this change of variables, the system takes the form

$$y_{n+2}y_n = \frac{\kappa(z_n - \lambda_1)(z_n - \lambda_2)}{(a_1\kappa z_n - q^n\lambda_2 b_0)(a_0 z_n - q^{n+2}\kappa\lambda_1 b_1)}, \tag{7.2a}$$

$$z_{n+2}z_n = \frac{\lambda_1\lambda_2\left(b_0 q^{n+2} y_{n+2} - 1\right)\left(b_1 q^{n+2} y_{n+2} - 1\right)}{(a_0 y_{n+2} - 1)(a_1 y_{n+2} - 1)}. \tag{7.2b}$$

This is the q-analogue of the sixth Painlevé equation [25].

8. General (s_1, s_2)-reduction

We have specified several cases of (2, 1)-reductions and a single case of a (2, 2)-reduction; however, this theory generalizes to an arbitrary (s_1, s_2)-reduction. Following [37], we let $s_1 = ag$ and $s_2 = bg$, with gcd(a, b)=1. Then we specify two integers, c and d, by

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1.$$

From this we define the variables

$$n = n(l, m) = \det\begin{pmatrix} a & b \\ l & m \end{pmatrix}, \qquad k = k(l, m) = \det\begin{pmatrix} l & m \\ c & d \end{pmatrix} \bmod g, \tag{8.1}$$

[8] A 2-integral is invariant under the second iterate of the map [16].

Table 1. A list of the Möbius point symmetries of the lattice equations that appear in the ABS list [3,4]. For Q4 we used the version given in [21].

ABS	Point symmetries
$H1$	$T_1 : w \to w + \lambda,\ T_2 : w \to \mu - w,$
$H3_{\delta=0}$	$T_1 : w \to \lambda w,\ T_2 : w \to \dfrac{\mu}{w},$
$H3_{\delta\neq 0}$	$T_1 : w \to -w,$
$Q1_{\delta\neq 0}$	$T_1 : w \to w + \lambda,\ T_2 : w \to \mu - w,$
$Q1_{\delta=0}$	$T_1 : w \to \dfrac{(\lambda_1\tau_1 - \lambda_2\tau_2)w - (\lambda_1 - \lambda_2)\tau_1\tau_2}{(\lambda_1 - \lambda_2)w + \lambda_2\tau_1 - \lambda_1\tau_2},$
$Q3_{\delta=0}$	$T_1 : w \to \lambda w,\ \ T_2 : w \to \mu/w,$
$Q3_{\delta\neq 0}$	$T_1 : w \to -w,$
$Q4$	$T_1 : w \to -w,\ T_2 : w \to 1/w,$
$A1_{\delta=0}$	$T_1 : w \to \lambda w,\ T_2 : w \to \mu/w,$
$A1_{\delta\neq 0}$	$T_1 : w \to -w,$

and

$$p = p(l, m) = \left\lfloor \frac{1}{g} \det \begin{pmatrix} l & m \\ c & d \end{pmatrix} \right\rfloor. \tag{8.2}$$

Now we perform the reduction in accordance with the rule

$$w_{l,m} \mapsto T^p u_n^k. \tag{8.3}$$

We note that the p variable is the power of the transformation, T, whereas the k is a superscript. The general (s_1, s_2)-reduction of (1.3) is given by the system of g equations:

$$Q(T^p u_n^k, T^{\tilde{p}} u_{n-b}^{k+d}, T^{\hat{p}} u_{n+a}^{k-c}, T^{\hat{\tilde{p}}} u_{n+a-b}^{k-c+d}; \alpha, \beta) = 0, \qquad k = 0, 1, \ldots, g-1, \tag{8.4}$$

where the superscripts are interpreted modulo g and $\tilde{p} = p(l+1, m)$ and $\hat{p} = p(l, m+1)$ are just the expressions for the p's shifted in the l and m directions, respectively. This choice of labels and powers of T ensures that any two ways of calculating an iterate, u_n^k, coincide due to the invariance of Q under the action of the twist, T.

We construct operators that govern the shifts $(l, m) \to (l + s_1, m + s_2)$ and $(l, m) \to (l + c, m + d)$, which have the effect

$$T\Psi_n = A_n \Psi_n, \tag{8.5a}$$

$$\Psi_{n+1} = B_n \Psi_n, \tag{8.5b}$$

in which the matrices, A_n and B_n, can be specified by

$$A_n \leftarrow \prod_{j=0}^{s_2-1} M_{l+s_1,m+j} \prod_{i=0}^{s_1-1} L_{l+i,m}, \tag{8.6a}$$

$$B_n \leftarrow \prod_{j=0}^{d-1} M_{l+c,m+j} \prod_{i=0}^{c-1} L_{l+i,m}, \tag{8.6b}$$

and n is given by (8.1), see also [35,36], where Lax matrices A_n and B_n are given in terms of a product along so called standard staircases. The determining equation (3.7) that defines the twist matrix is also a valid ansatz for the general (s_1, s_2)-reduction.

A list of possible (Möbius) twists for the equations in the ABS-list appears in table 1.

Let us conclude by mentioning twisted reductions for non-autonomous multiplicative equations, i.e. those for which Q and the Lax matrices depend on α/β only. Under this assumption[9] the reduction is consistent, provided $\alpha_{l+s_1}/\beta_{m+s_2} = \alpha_l/\beta_m$. By separation of variables this gives

$$\frac{\alpha_{l+s_1}}{\alpha_l} = \frac{\beta_{m+s_2}}{\beta_m} := q^{abg},$$

which is solved by

$$\alpha_l = a_{l \bmod s_1} q^{bl}, \qquad \beta_m = b_{m \bmod s_2} q^{am}. \tag{8.7}$$

A simple choice of spectral variable is $x = q^l$, in which the product representations of A_n and B_n, given above by (8.6*a*) and (8.6*b*), depend on x, giving $A_n(x)$ and $B_n(x)$. These matrices define a linear system

$$TY_n(q^{abg}x) = A_n(x)Y_n(x), \tag{8.8a}$$

$$Y_{n+1}(q^{cb}x) = B_n(x)Y_n(x), \tag{8.8b}$$

along with the definition of the twist matrix (3.14) gives the compatibility

$$S_{n+1}^{-1}A_{n+1}(q^{cb}x)B_n(x) - B_n(q^{abg}x)S_n^{-1}A_n(x) = 0. \tag{8.9}$$

This compatibility is equivalent to the system of g equations that define the non-autonomous reductions,

$$Q(T^p u_n^k, T^{\tilde{p}} u_{n-b}^{k+d}, T^{\hat{p}} u_{n+a}^{k-c}, T^{\hat{\tilde{p}}} u_{n+a-b}^{k-c+d}; \alpha_l/\beta_m) = 0, \qquad k = 0, 1, \ldots, g-1, \tag{8.10}$$

where we should recall that α_l/β_m, as a function of n, is

$$\frac{\alpha_l}{\beta_m} = \frac{a_{l \bmod s_1}}{b_{m \bmod s_2}} q^{-n}.$$

This provides a Lax representation for the twisted (s_1, s_2)-reduction with general s_1 and s_2.

9. Conclusions

We have presented a generalization of periodic reductions, that would appear to be new. Applying this to integrable equations, the resulting reductions possess Lax representations. This method can be used to obtain many additional integrable mappings. This can be done either by considering other reductions or by starting from other integrable equations on quads (both of ABS type or non-ABS type), or from (multi-component) equations on other stencils. The method proposed in this paper seems analogous to Sklyanin's method for generalizing periodic boundary conditions for integrable quantum systems [54]. Finally, we note that twisted reductions may also apply to non-integrable equations (although in that case there will be no Lax representations).

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[9] The case for additive type twisted reductions can be formulated analogously to what is presented here.

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