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Symbolic Computation of Lax Pairs of Partial Difference Equations using Consistency Around the Cube

T. Bridgman · W. Hereman · G.R.W. Quispel · P.H. van der Kamp

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Abstract A three-step method due to Nijhoff and Bobenko & Suris to derive a Lax pair for scalar partial difference equations (P∆Es) is reviewed. The method assumes that the P∆Es are defined on a quadrilateral, and consistent around the cube. Next, the method is extended to systems of P∆Es where one has to carefully account for equations defined on edges of the quadrilateral. Lax pairs are presented for scalar integrable P∆Es classified by Adler, Bobenko, and Suris and systems of P∆Es including the integrable two-component potential Korteweg–de Vries lattice system, as well as nonlinear Schrödinger and Boussinesq-type lattice systems. Previously unknown Lax pairs are presented for P∆Es recently derived by Hietarinta (J. Phys. A, Math. Theor. 44:165204, 2011). The method is algorithmic and is being implemented in MATHEMATICA.

Keywords Integrable lattice equations · Lax pairs · Consistency around the cube

Mathematics Subject Classification 37K10 · 39A10 · 82B20

At the occasion of his 60th birthday, we like to dedicate this paper to Peter Olver, whose work has inspired us throughout our careers.

Communicated by Elizabeth Mansfield.

T. Bridgman (✉) · W. Hereman
Department of Applied Mathematics and Statistics, Colorado School of Mines, Golden, CO 80401-1887, USA
e-mail: tbridgma@mines.edu

G.R.W. Quispel · P.H. van der Kamp
Department of Mathematics and Statistics, La Trobe University, Melbourne, Victoria 3086, Australia

1 Introduction

The original Lax pair [11] was a duo of commuting linear differential operators representing the integrable Korteweg–de Vries (KdV) equation. Lax's idea was to replace a nonlinear partial differential equation (PDE), such as the KdV equation, by a pair of *linear* PDEs of high-order (in an auxiliary eigenfunction) whose compatibility requires that the *nonlinear* PDE holds. One can write these high-order linear PDEs as a system of PDEs of first order; hence, replacing the Lax operators with a pair of matrices. The Lax equation to be satisfied by these matrices is commonly referred to as the zero-curvature representation [34] of the nonlinear PDE. The discovery of Lax pairs was crucial for the further development of the inverse scattering transform (IST) method, which had been introduced in [8].

For partial difference equations (PΔEs) Lax pairs first appeared in the work of Ablowitz and Ladik [1, 2], and subsequently in [16] for other equations. The fundamental characterization of *integrable* PΔEs as being multi-dimensionally consistent [6, 14] is intimately related to the existence of a Lax pair.

Lax pairs for PΔEs are not only crucial for applying the IST, they can be used to construct integrals for mappings and correspondences obtained as periodic reductions, using the so-called staircase method. This method was developed in [18] and extended in [19] to cover more general reductions. Essential to the staircase method is the construction of a product of Lax matrices (the *monodromy matrix*) whose characteristic polynomial is an invariant of the evolution. In fact, the monodromy matrix can be interpreted as one of the Lax matrices for the reduced mapping [20–22]. Through expansion of the characteristic equation of the monodromy matrix in the spectral parameters a number of functionally independent invariants can be obtained. A recent investigation [27] supports the idea that the staircase method provides sufficiently many integrals for the periodic reductions to be completely integrable (in the sense of Liouville–Arnold).

Finding a Lax pair for a given nonlinear equation, whether continuous or discrete, is generally a difficult task. For PDEs the theory of pseudo potentials [28] might lead to a Lax pair, but it only works in certain cases. The most powerful method to find Lax pairs is the dressing method developed by Zakharov and Shabat in 1974 (see, e.g., [33]). Building on the key idea of the dressing method, there exists a straightforward, algorithmic approach to derive a Lax pair [6, 14] for scalar PΔEs that are *consistent around the cube* (CAC). That approach is reviewed in Sect. 2.1. In Sect. 3, it is applied to systems of lattice equations, as was done in [24, 29] for the case of the Boussinesq system.

We are currently developing MATHEMATICA software for the symbolic computation of Lax pairs for lattice equations [7, 9]. Section 4 outlines the implementation strategy for the verification of the CAC property and the computation (and subsequent verification) of the Lax pair. With the exception of the Q_4 equation whose Lax pair was given in [14], the software has been used to produce Lax pairs of the ABS equations [3] and the (α, β)-equation. The latter is also known as the NQC equation after Nijhoff, Quispel, and Capel [16] and its Lax pair was first reported in [25].

With respect to lattice systems, we computed Lax pairs of the Boussinesq and Toda-modified Boussinesq systems [15], as well as the Schwarzian Boussinesq [13] and modified Boussinesq [32] systems. Using the code, we also computed Lax pairs for the two-component potential KdV and nonlinear Schrödinger systems [12, 31]. Details of the calculations, and alternative Lax pairs, are given in Sect. 5. We obtained new Lax pairs for the two- and three-component Hietarinta systems [10]. In contrast to the 4×4 Lax matrices for the Hietarinta systems [10] obtained (independently) in [35], the Lax matrices presented in this paper are 3×3 matrices.

2 Scalar Partial Difference Equations

2.1 Consistency Around the Cube for Scalar PΔEs

The concept of multi-dimensional consistency was introduced independently in [6, 17]. The key idea is to embed the equation consistently into a multi-dimensional lattice by imposing copies of the same equation, albeit with different lattice parameters in different directions. The consistency for embedding a two-dimensional lattice equation, defined on an elementary quadrilateral, into a three-dimensional lattice on a cube is commonly referred to as consistency around the cube (CAC). For multi-affine nonlinear PΔEs with the CAC property there is an algorithmic way of deriving a Lax pair.

In this paper we consider PΔEs,

$$\mathcal{F}(x, x_1, x_2, x_{12}; p, q) = 0, \tag{1}$$

which are defined on a two-dimensional quad-graph as shown in Fig. 1. The field variable $x = x_{n,m}$ depends on lattice variables n and m. A shift of x in the horizontal direction (the 1-direction) is denoted by $x_1 \equiv x_{n+1,m}$. A shift in the vertical or 2-direction by $x_2 \equiv x_{n,m+1}$ and a shift in both directions by $x_{12} \equiv x_{n+1,m+1}$. Furthermore, $\mathcal{F}$ depends on the lattice parameters p and q which correspond to the edges of the quadrilateral. Alternate notations are used in the literature. For instance, many authors denote (x, x_1, x_2, x_{12}) by $(x, \tilde{x}, \hat{x}, \hat{\tilde{x}})$ while others use $(x_{00}, x_{10}, x_{01}, x_{11})$.

In this paper, we assume that the initial values (indicated by solid circles) for x, x_1 and x_2 can be specified and that the value of x_{12} (indicated by an open circle) can be uniquely determined by (1). To have single-valued maps, we assume that $\mathcal{F}$ is multi-affine [6], which is sometimes called multi-linear. Atkinson [4] and Atkinson and Nieszporksi [5] have recently given examples of PΔEs that are multi-*quadratic* and multi-dimensionally consistent.

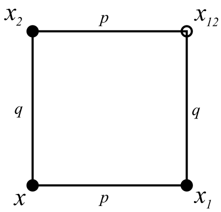
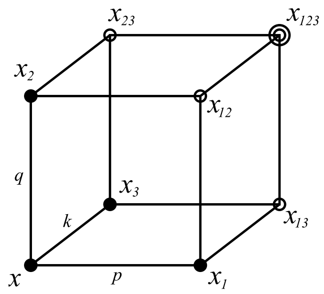


Fig. 1 The PΔE is defined on the simplest quadrilateral (a square)

Fig. 2 The PΔE holds on each face of the cube.



In the simplest case, $\mathcal{F}$ is a *scalar* relation between values of a single dependent variable x and its shifts (located at the vertices of an elementary square). Nonlinear lattice equations of type (1) arise, for example, as the permutability condition for Bäcklund transformations associated with integrable partial differential equations (PDEs).

In more complicated cases, $\mathcal{F}$ is a nonlinear *vector* function of the vector $\mathbf{x}$ with several components. In that case, (1) represents a system of PΔEs. These systems are called multi-component lattice equations. In such systems some equations might only be defined on the edges of the square while others are defined on the whole square. The vector case will be considered in Sect. 3.

To arrive at a cube, the planar quadrilateral is extended into the third dimension as shown in Fig. 2, where parameter k is the lattice parameter in the third direction. Although not explicitly shown in Fig. 2, all parallel edges carry the same lattice parameters.

A key assumption is that the original equation(s) holds on all faces of the cube. These equations can therefore be generated by changes of variables and parameters, or shifts of the original PΔE. On the cube, they can be visualized as either translations, or rotations of the faces. For example, the equation on the left face can be obtained via a rotation of the front face along the vertical axis connecting x and x_2. This amounts to applying to (1) the substitutions

$$x_1 \to x_3, \quad x_{12} \to x_{23}, \quad \text{and} \quad p \to k, \tag{2}$$

yielding $\mathcal{F}(x, x_3, x_2, x_{23}; k, q) = 0$. The equation on the *back* face of the cube can be generated via a shift of (1) in the third direction, letting

$$x \to x_3, \quad x_1 \to x_{13}, \quad x_2 \to x_{23}, \quad \text{and} \quad x_{12} \to x_{123}, \tag{3}$$

which yields $\mathcal{F}(x_3, x_{13}, x_{23}, x_{123}; p, q) = 0$.

The equations on the *back*, *right*, and *top* faces of the cube all involve the unknown x_{123} (indicated by the double open circle). Solving them yields three expressions for x_{123}. Consistency around the cube of the PΔE requires that one can uniquely determine x_{123} and that all three expressions coincide. As discussed in [23], this three-dimensional consistency establishes integrability.

The consistency property does not depend on the actual mappings used to generate the PΔEs on the various faces of the cube. Mappings such as (2) and (3), which express the symmetries of the PΔEs are merely a tool for generating the needed PΔEs quickly.

Example 1 Consider the lattice modified KdV (mKdV) equation [23] (also classified as H_3 with $\delta = 0$ as listed in Table 1),

$$p(xx_1 + x_2x_{12}) - q(xx_2 + x_1x_{12}) = 0. \tag{4}$$

This equation is defined on the *front* face of the cube. To verify CAC, variations of the original PΔE on the *left* and *bottom* faces of the cube are generated. Hence, (4) is supplemented with two additional equations:

$$p(xx_3 + x_2x_{23}) - q(xx_2 + x_3x_{23}) = 0, \tag{5a}$$

$$p(xx_1 + x_3x_{13}) - q(xx_3 + x_1x_{13}) = 0, \tag{5b}$$

which yield solutions for x_{12}, x_{13}, and x_{23}:

$$x_{12} = \frac{x(px_1 - qx_2)}{qx_1 - px_2}, \tag{6a}$$

$$x_{13} = \frac{x(px_1 - kx_3)}{kx_1 - px_3}, \tag{6b}$$

$$x_{23} = \frac{x(qx_2 - kx_3)}{kx_2 - qx_3}. \tag{6c}$$

Equations for the remaining faces (i.e., *back, right* and *top*) are then generated:

$$p(x_3x_{13} + x_{23}x_{123}) - q(x_3x_{23} + x_{13}x_{123}) = 0, \tag{7a}$$

$$p(x_1x_{13} + x_{12}x_{123}) - q(x_1x_{12} + x_{13}x_{123}) = 0, \tag{7b}$$

$$p(x_2x_{12} + x_{23}x_{123}) - q(x_2x_{23} + x_{12}x_{123}) = 0. \tag{7c}$$

Each of these reference x_{123} and thus yield three distinct solutions for x_{123},

$$x_{123} = \frac{x_3(px_{13} - qx_{23})}{qx_{13} - px_{23}}, \tag{8a}$$

$$x_{123} = \frac{x_2(px_{12} - kx_{23})}{kx_{12} - px_{23}}, \tag{8b}$$

$$x_{123} = \frac{x_1(qx_{12} - kx_{13})}{kx_{12} - qx_{13}}. \tag{8c}$$

Remarkably, after substitution of (6a)–(6c) into (8a)–(8c) one arrives at the *same expression* for x_{123}, namely,

$$x_{123} = -\frac{px_2x_3(k^2 - q^2) + qx_1x_3(p^2 - k^2) + kx_1x_2(q^2 - p^2)}{px_1(k^2 - q^2) + qx_2(p^2 - k^2) + kx_3(q^2 - p^2)}. \tag{9}$$

Table 1 Lax pairs of scalar P∆Es

Name	Equation	Matrix L	Alternate t values	Ref.
H_1	$(x-x_{12})(x_1-x_2)+q-p=0$	$t\begin{bmatrix} x & p-k-xx_1 \\ 1 & -x_1 \end{bmatrix}$ with $t=1$		[3]
H_2	$(x-x_{12})(x_1-x_2)+(q-p)(x+x_1+x_2+x_{12}+p+q)=0$	$t\begin{bmatrix} p-k+x & p^2-k^2+(p-k)(x+x_1)-xx_1 \\ 1 & -(p-k+x_1) \end{bmatrix}$ with $t=\frac{1}{\sqrt{p+x+x_1}}$		[3]
H_3	$p(xx_1+x_2x_{12})-q(xx_2+x_1x_{12})+\delta(p^2-q^2)=0$	$t\begin{bmatrix} kx & -\delta(p^2-k^2)-pxx_1 \\ p & -kx_1 \end{bmatrix}$ with $t=\frac{1}{\sqrt{\delta p+xx_1}}$		[3]
A_1	$(x+x_2)(x_1+x_{12})-q(x+x_1)(x_2+x_{12})-\delta^2pq(p-q)=0$	$t\begin{bmatrix} kx+\eta x_1 & -p(xx_1+\delta^2k\eta) \\ p & -(kx_1+\eta x) \end{bmatrix}$ where $\eta=k-p$, with $t=\frac{1}{\sqrt{(\delta p-(x+x_1))(\delta p+(x+x_1))}}$		[3]
A_2	$(q^2-p^2)(xx_1x_2x_{12}+1)+q(p^2-1)(xx_2+x_1x_{12})-p(q^2-1)(xx_1+x_2x_{12})=0$	$t\begin{bmatrix} k\gamma x & -(\tau+p\sigma xx_1) \\ p\sigma+\tau xx_1 & -k\gamma x_1 \end{bmatrix}$ where $\gamma=p^2-1,\ \sigma=k^2-1$ and $\tau=p^2-k^2$ with $t=\frac{1}{\sqrt{-(p-xx_1)(pxx_1-1)}}$		[3]

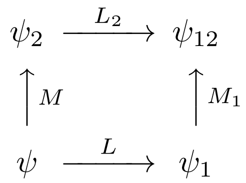


Fig. 3 Commuting scheme resulting in the Lax equation. M_1 denotes the shift of M in the 1-direction (*horizontally*). L_2 denotes the shift of L in the 2-direction (*vertically*)

Thus, (4) is consistent around the cube. The consistency is apparent from the following symmetry of the right hand side of (9). If we replace the lattice parameters (p, q, k) by (l_1, l_2, l_3) the expression would be invariant under any permutation of the indices $\{1, 2, 3\}$.

Additionally, (9) does not reference x. This independence is referred to as the *tetrahedron property*. Indeed, through (9), the top of a tetrahedron (located at x_{123}) is connected to the base of the tetrahedron with corners at x_1, x_2 and x_3.

2.2 Computation of Lax Pairs for Scalar PΔEs

In analogy with the definition of Lax pairs (in matrix form) for PDEs, a Lax pair for a PΔE is a pair of matrices, (L, M), such that the compatibility of the linear equations, for an auxiliary vector function ψ,

$$\psi_1 = L\psi, \tag{10a}$$

$$\psi_2 = M\psi, \tag{10b}$$

is equivalent to the PΔE. The crux is to find suitable matrices L and M so that the nonlinear PΔE can be replaced by (10a)–(10b). To avoid trivial cases, the compatibility of (10a) and (10b) should only hold on solutions of the given nonlinear PΔE.

The compatibility of (10a) and (10b) can be readily expressed as follows. Shift (10a) in the 2-direction, i.e., $\psi_{12} = L_2\psi_2 = L_2 M\psi$. Shift (10b) in the 1-direction, i.e., $\psi_{21} = \psi_{12} = M_1\psi_1 = M_1 L\psi$, and equate the results. Hence, $L_2 M\psi = M_1 L\psi$ must hold on solutions of the PΔE. The compatibility is visualized in Fig. 3, where commutation of the scheme indeed requires that $L_2 M = M_1 L$. The corresponding Lax equation is thus

$$L_2 M - M_1 L \doteq 0, \tag{11}$$

where $\doteq$ denotes that the equation holds for solutions of the PΔE.

As is the case for completely integrable PDEs, Lax pairs of PΔEs are not unique for they are equivalent under gauge transformations. Specifically, if (L, M) is a Lax pair then so is $(\mathcal{L}, \mathcal{M})$ where

$$\mathcal{L} = \mathcal{G}_1 L \mathcal{G}^{-1}, \qquad \mathcal{M} = \mathcal{G}_2 M \mathcal{G}^{-1}, \tag{12}$$

for any arbitrary non-singular matrix $\mathcal{G}$. Indeed, $(\mathcal{L}, \mathcal{M})$ satisfy $\mathcal{L}_2\mathcal{M} - \mathcal{M}_1\mathcal{L} \doteq 0$, which follows from (11) by pre-multiplication by $\mathcal{G}_{12}$ and post-multiplication by $\mathcal{G}^{-1}$. Alternatively, $\phi_1 = \mathcal{L}\phi$ and $\phi_2 = \mathcal{M}\phi$, provided $\phi = \mathcal{G}\psi$. The Lax pairs (L, M) and $(\mathcal{L}, \mathcal{M})$ are said to be *gauge equivalent*.

Returning to Example 1, we show that the CAC property implicitly determines the Lax pair of a PΔE. Indeed, observe that, as a consequence of the multi-affine structure of the original PΔE, the numerator and denominator of x_{13} in (6b) are linear in x_3. In analogy with the linearization of Riccati equations, substitute $x_3 = \frac{f}{F}$ into (6b), yielding

$$x_{13} = \frac{f_1}{F_1} = \frac{fkx - Fpxx_1}{fp - Fkx_1}. \tag{13}$$

Hence,

$$f_1 = t(fkx - Fpxx_1) \tag{14}$$

and

$$F_1 = t(fp - Fkx_1), \tag{15}$$

where $t(x, x_1; p, k)$ is a function still to be determined. Defining $\psi = \left[\begin{smallmatrix} f \\ F \end{smallmatrix}\right]$, system (14)–(15) can be written in matrix form (10a) where $L = tL_c$ and the "core" of the Lax matrix L is given by

$$L_c = \begin{bmatrix} kx & -pxx_1 \\ p & -kx_1 \end{bmatrix}. \tag{16}$$

Using (6c), the computation of the Lax matrix M proceeds analogously. Indeed,

$$x_{23} = \frac{f_2}{F_2} = \frac{fkx - Fqxx_2}{fq - Fkx_2} \tag{17}$$

holds if $f_2 = s(fkx - Fqxx_2)$ and $F_2 = s(fq - Fkx_2)$ where $s(x, x_2; q, k)$ is a common factor to be determined. Thus, we obtain (10b) where $M = sM_c$ with

$$M_c = \begin{bmatrix} kx & -qxx_2 \\ q & -kx_2 \end{bmatrix}. \tag{18}$$

Note that x_{23} can be obtained from x_{13}, and hence M_c from L_c, by replacing $x_1 \to x_2$ (or simply, $1 \to 2$) and $p \to q$. The final step is to compute s and t.

2.3 Determination of the Scalar Factors for Scalar PΔEs

Specific values for s and t can be computed using (11). Substituting $L = tL_c$ and $M = sM_c$ yields

$$st_2(L_c)_2 M_c - ts_1(M_c)_1 L_c \doteq 0. \tag{19}$$

All elements in the matrix on the left hand side must vanish. Remarkably, this yields a unique expression for the ratio $\frac{st_2}{ts_1}$.

For Example 1, using (16) and (18), Eq. (19) reduces to

$$\left(\frac{xx_1ts_1 - xx_2st_2}{px_2 - qx_1}\right) \times \begin{bmatrix} (k^2-p^2)qx_1 - (k^2-q^2)px_2 & k(p^2-q^2)x_1x_2 \\ -k(p^2-q^2) & (k^2-p^2)qx_1 - (k^2-q^2)px_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{20}$$

This requires that

$$\frac{st_2}{ts_1} \doteq \frac{x_1}{x_2}, \tag{21}$$

which has an infinite family of solutions. Indeed, the left hand side of (21) is invariant under the change

$$t \to \frac{a_1}{a} t, \qquad s \to \frac{a_2}{a} s, \tag{22}$$

where $a(x)$ is arbitrary. Consistent with the notations in Sect. 2, a_1 and a_2 denote the shifts of a in the 1- and 2-direction, respectively. By inspection,

$$t = s = \frac{1}{x} \tag{23}$$

and

$$t = \frac{1}{x_1}, \qquad s = \frac{1}{x_2}; \tag{24}$$

both satisfy (21). Note that (23) can be mapped into (24) by taking $a = 1/x$.

Avoiding guess work, t and s can be computed by taking the determinant of (19). If L_c and M_c are $n \times n$ matrices, then

$$(st_2)^n \det (L_c)_2 \det M_c = (ts_1)^n \det (M_c)_1 \det L_c, \tag{25}$$

yielding

$$\frac{st_2}{ts_1} = \sqrt[n]{\frac{\det(M_c)_1 \det L_c}{\det(L_c)_2 \det M_c}}, \tag{26}$$

which is satisfied by

$$t = \frac{1}{\sqrt[n]{\det L_c}}, \qquad s = \frac{1}{\sqrt[n]{\det M_c}}. \tag{27}$$

For Example 1, i.e., Eq. (4), by substituting (16) and (18) into (26), one then obtains

$$t = \frac{1}{\sqrt{(p^2 - k^2)xx_1}}, \qquad s = \frac{1}{\sqrt{(q^2 - k^2)xx_2}}. \tag{28}$$

The constant factors involving p, q and k are irrelevant. Therefore, (28) can be replaced by

$$t = \frac{1}{\sqrt{xx_1}}, \qquad s = \frac{1}{\sqrt{xx_2}}. \tag{29}$$

Thus, using the *determinant method*, a Lax pair for (4) is

$$L = \frac{1}{\sqrt{xx_1}} \begin{bmatrix} kx & -pxx_1 \\ p & -kx_1 \end{bmatrix}, \qquad M = \frac{1}{\sqrt{xx_2}} \begin{bmatrix} kx & -qxx_2 \\ q & -kx_2 \end{bmatrix}. \tag{30}$$

The irrational t and s in (29) can be transformed into (23), by taking $a = \sqrt{x}$, or into (24), by $a = \frac{1}{\sqrt{x}}$, both yielding rational Lax pairs.

3 Systems of Partial Difference Equations

Section 2 dealt with single (scalar) PΔEs, i.e., equations involving only one field variable (denoted by x). This section covers systems of PΔEs defined on quadrilaterals involving multiple field variables. Here we will consider examples involving three field variables x, y, and z. Figures 1 and 2 still apply provided we replace the scalar x by vector $\mathbf{x} \equiv (x, y, z)$. Hence, $\mathbf{x}_1 = (x_1, y_1, z_1)$, $\mathbf{x}_2 = (x_2, y_2, z_2)$, $\mathbf{x}_{12} = (x_{12}, y_{12}, z_{12})$, etc.

3.1 Consistency Around the Cube for Systems of PΔEs

To apply the algorithm in Sect. 2.2 to systems of PΔEs, it is necessary to maintain consistency for all equations on all six faces of the cube, handle the edge equations in an appropriate way, and ultimately arrive at the same expressions for x_{123}, as well as for y_{123} and z_{123}.

Example 2 Consider the lattice Schwarzian Boussinesq system [13]:

$$x_1 y - z_1 + z = 0, \tag{31a}$$

$$x_2 y - z_2 + z = 0, \tag{31b}$$

$$x y_{12}(y_1 - y_2) - y(p x_1 y_2 - q x_2 y_1) = 0. \tag{31c}$$

Equations (31a) and (31b) are defined along a single edge of the square while (31c) is defined on the whole square. The edge equations, unlike the face equation, can be shifted in the 1- or 2-directions while still remaining on the square. Then, (31a)–(31c) is augmented with additional shifted edge equations,

$$x_{12} y_2 - z_{12} + z_2 = 0, \tag{32a}$$

$$x_{12} y_1 - z_{12} + z_1 = 0, \tag{32b}$$

obtained from (31a) and (31b), respectively. Solving for the variables $\mathbf{x}_{12} = (x_{12}, y_{12}, z_{12})$ referenced in the augmented system (i.e., (31a)–(31c) augmented with (32a), (32b)) gives

$$x_{12} = \frac{z_2 - z_1}{y_1 - y_2}, \tag{33a}$$

$$y_{12} = \frac{y(p x_1 y_2 - q x_2 y_1)}{x(y_1 - y_2)}, \tag{33b}$$

$$z_{12} = \frac{y_1 z_2 - y_2 z_1}{y_1 - y_2}. \tag{33c}$$

Continuing as before by generating the variations of (31a)–(31c) on the faces of the cube and solving for the variables with double subscripts yields $\mathbf{x}_{13}$ and $\mathbf{x}_{23}$. Indeed, from the equations on the *bottom* face (not shown) one gets $\mathbf{x}_{13}$ with components

$$x_{13} = \frac{z_3 - z_1}{y_1 - y_3}, \tag{34a}$$

$$y_{13} = \frac{y(px_1y_3 - kx_3y_1)}{x(y_1 - y_3)}, \tag{34b}$$

$$z_{13} = \frac{y_1z_3 - y_3z_1}{y_1 - y_3}, \tag{34c}$$

which readily follow from (33a)–(33c) by replacing $\mathbf{x}_2 \to \mathbf{x}_3, \mathbf{x}_{12} \to \mathbf{x}_{13}$, and $q \to k$. Or simpler, $2 \to 3$ and $q \to k$. Similarly, the equations on the *left* face of the cube determine $\mathbf{x}_{23}$ with components

$$x_{23} = \frac{z_2 - z_3}{y_3 - y_2}, \tag{35a}$$

$$y_{23} = \frac{y(kx_3y_2 - qx_2y_3)}{x(y_3 - y_2)}, \tag{35b}$$

$$z_{23} = \frac{y_3z_2 - y_2z_3}{y_3 - y_2}, \tag{35c}$$

easily obtained by a change of labels and parameters, namely, $1 \to 2$, $p \to q$, $2 \to 3$, and $q \to k$. Likewise, the equations on the *back* face (not shown) determine $\mathbf{x}_{123}$ with components

$$x_{123} = \frac{z_{23} - z_{13}}{y_{13} - y_{23}}, \tag{36a}$$

$$y_{123} = \frac{y_3(px_{13}y_{23} - qx_{23}y_{13})}{x_3(y_{13} - y_{23})}, \tag{36b}$$

$$z_{123} = \frac{y_{13}z_{23} - y_{23}z_{13}}{y_{13} - y_{23}}, \tag{36c}$$

which follow from (33a)–(33c) by applying the shift in the third direction. This amounts to "adding" a label 3 to all variables. Similarly, the equations on the *right* face (suppressed) yield $\mathbf{x}_{123}$ with components

$$x_{123} = \frac{z_{12} - z_{13}}{y_{13} - y_{12}}, \tag{37a}$$

$$y_{123} = \frac{y_1(kx_{13}y_{12} - qx_{12}y_{13})}{x_1(y_{13} - y_{12})}, \tag{37b}$$

$$z_{123} = \frac{y_{13}z_{23} - y_{12}z_{13}}{y_{13} - y_{12}}, \tag{37c}$$

which follow from (35a)–(35c) by applying a shift in the 1-direction. Finally, the equations on the *top* face (suppressed) yield

$$x_{123} = \frac{z_{23} - z_{12}}{y_{12} - y_{23}}, \tag{38a}$$

$$y_{123} = \frac{y_2(px_{12}y_{23} - kx_{23}y_{12})}{x_2(y_{12} - y_{23})}, \tag{38b}$$

$$z_{123} = \frac{y_{12}z_{23} - y_{23}z_{12}}{y_{12} - y_{23}}, \tag{38c}$$

obtained from (34a)–(34c) by a shift in the 2-direction.

Using (33a)–(35c) to evaluate the expressions (36a)–(38c) yields the *same* $\mathbf{x}_{123}$ with

$$x_{123} = \frac{x(x_1 - x_2)(y_1(z_2 - z_3) + y_2(z_3 - z_1) + y_3(z_1 - z_2))}{(z_1 - z_2)(px_1(y_3 - y_2) + qx_2(y_1 - y_3) + kx_3(y_2 - y_1))}, \tag{39a}$$

$$y_{123} = \frac{q(z_2 - z_1)(kx_3y_1 - px_1y_3) + k(z_3 - z_1)(px_1y_2 - qx_2y_1)}{x_1(px_1(y_3 - y_2) + qx_2(y_1 - y_3) + kx_3(y_2 - y_1))}, \tag{39b}$$

$$z_{123} = \frac{px_1(y_3z_2 - y_2z_3) + qx_2(y_1z_3 - y_3z_1) + kx_3(y_2z_1 - y_1z_2)}{px_1(y_3 - y_2) + qx_2(y_1 - y_3) + kx_3(y_2 - y_1)}. \tag{39c}$$

Thus, (31a)–(31c) is *multi-dimensionally* consistent around the cube, i.e., the systems of PΔEs is consistent around the cube with respect to each component of $\mathbf{x}$, i.e., x, y and z.

The expressions for x_{123} and y_{123} can be written in more symmetric form by eliminating z_1, z_2, and z_3. To do so, we use the edge equations

$$x_3y - z_3 + z = 0, \tag{40a}$$

$$x_2y - z_2 + z = 0, \tag{40b}$$

defined on the *left* face of the cube. Subtracting (31a) from (31b) and (40a) from (40b) yields

$$\frac{z_2 - z_1}{x_2 - x_1} = \frac{z_3 - z_2}{x_3 - x_2} = \frac{z_3 - z_1}{x_3 - x_1} = y. \tag{41}$$

Using the above ratios, (39a) and (39b) can be replaced by

$$x_{123} = \frac{x(y_1(x_2 - x_3) + y_2(x_3 - x_1) + y_3(x_1 - x_2))}{px_1(y_3 - y_2) + qx_2(y_1 - y_3) + kx_3(y_2 - y_1)}, \tag{42a}$$

$$y_{123} = \frac{y(kqy_1(x_2 - x_3) + kpy_2(x_3 - x_1) + pqy_3(x_1 - x_2))}{px_1(y_3 - y_2) + qx_2(y_1 - y_3) + kx_3(y_2 - y_1)}. \tag{42b}$$

Before continuing with the calculations of a Lax pair, it is worth noting that (31a)–(31c) does not satisfy the tetrahedron property because x explicitly appears in the right hand side of (39a). The impact of not having the tetrahedron property remains unclear but does not affect the computation of a Lax pair.

3.2 Computation of a Lax Pair for Systems of PΔEs

Both the numerators and denominators of the components of $\mathbf{x}_{13}$ and $\mathbf{x}_{23}$ (in (34a)–(34c) and (35a)–(35c), respectively), are affine linear in the components of $\mathbf{x}$. Due to their linearity in x_3, y_3 and z_3, substitution of fractional expressions for x_3, y_3 and z_3 will allow one to compute Lax matrices. In contrast to the scalar case, the computations are more subtle because the edge equations on the left face of the cube introduce constraints between x_3 and z_3.

Continuing with Example 2, solving (40a) for x_3 yields

$$x_3 = \frac{z_3 - z}{y}. \tag{43}$$

Therefore, setting

$$z_3 = \frac{f}{F} \tag{44a}$$

and

$$y_3 = \frac{g}{G} \tag{44b}$$

determines

$$x_3 = \frac{z_3 - z}{y} = \frac{f - Fz}{Fy}. \tag{44c}$$

Substituting (44a)–(44c) into (34a)–(34c) then yields

$$x_{13} = \frac{G(Fz_1 - f)}{F(g - Gy_1)}, \tag{45a}$$

$$y_{13} = \frac{Gfky_1 - Fgx_1y - FGky_1z}{Fx(g - Gy_1)}, \tag{45b}$$

$$z_{13} = \frac{Fgz_1 - Gfy_1}{F(g - Gy_1)}, \tag{45c}$$

which are not yet linear in f, g, F, and G. Additional constraints between f, g, F and G will achieve this goal. Indeed, setting $G = F$ simplifies (45a)–(45c) into

$$x_{13} = \frac{f - Fz_1}{Fy_1 - g}, \tag{46a}$$

$$y_{13} = \frac{gpx_1y - fky_1 + Fky_1z}{x(Fy_1 - g)}, \tag{46b}$$

$$z_{13} = \frac{fy_1 - gz_1}{Fy_1 - g}. \tag{46c}$$

Simultaneously, (44a)–(44c) reduces to

$$z_3 = \frac{f}{F}, \tag{47a}$$

$$y_3 = \frac{g}{F}, \tag{47b}$$

$$x_3 = \frac{f - Fz}{Fy}, \tag{47c}$$

whose shifts in the 1-direction must be compatible with (46a)–(46c). Equating $z_{13} = \frac{f_1}{F_1}$ with (46c) requires that

$$f_1 = t(fy_1 - gz_1) \tag{48}$$

and

$$F_1 = t(Fy_1 - g). \tag{49}$$

Next, equating $y_{13} = \frac{g_1}{F_1}$ with (46b) gives

$$g_1 = t\frac{1}{x}(gpx_1y - fky_1 + Fky_1z). \tag{50}$$

Finally, one has to verify that the 1-shift of (47c),

$$x_{13} = \frac{f_1 - F_1z_1}{F_1y_1}, \tag{51}$$

matches (46a). That is indeed the case. After substitution of f_1 and F_1 into (51)

$$x_{13} = \frac{t(fy_1 - gz_1) - t(Fy_1 - g)z_1}{t(Fy_1 - g)y_1} = \frac{f - Fz_1}{Fy_1 - g}. \tag{52}$$

Defining $\psi = \left[\begin{smallmatrix} g \\ f \\ F \end{smallmatrix}\right]$, Eqs. (48)–(50) can be written in matrix form yielding (10a) with

$$L = t\begin{bmatrix} \frac{px_1y}{x} & -\frac{ky_1}{x} & \frac{ky_1z}{x} \\ -z_1 & y_1 & 0 \\ -1 & 0 & y_1 \end{bmatrix}, \tag{53}$$

where $t(\mathbf{x}, \mathbf{x}_1; p, k)$. Similarly, from (35a)–(35c) one derives

$$M = s\begin{bmatrix} \frac{qx_2y}{x} & -\frac{ky_2}{x} & \frac{ky_2z}{x} \\ -z_2 & y_2 & 0 \\ -1 & 0 & y_2 \end{bmatrix}, \tag{54}$$

which can also be obtained from (53) by applying the replacement rules $1 \to 2$ and $p \to q$.

3.3 Determination of the Scalar Factors for Systems of P∆Es

As discussed in Sect. 2.3, specific values for s and t may be computed algorithmically using (27). For Example 2, this yields

$$t = \frac{1}{\sqrt[3]{\frac{(k-p)y_1^2(z-z_1)}{x}}}, \qquad s = \frac{1}{\sqrt[3]{\frac{(k-q)y_2^2(z-z_2)}{x}}}. \tag{55}$$

Canceling trivial factors, a Lax pair for (31a)–(31c) is thus given by

$$L = \sqrt[3]{\frac{x}{y_1^2(z - z_1)}}\begin{bmatrix} \frac{px_1y}{x} & -\frac{ky_1}{x} & \frac{ky_1z}{x} \\ -z_1 & y_1 & 0 \\ -1 & 0 & y_1 \end{bmatrix}, \tag{56a}$$

$$M = \sqrt[3]{\frac{x}{y_2^2(z - z_2)}} \begin{bmatrix} \frac{qx_2y}{x} & -\frac{ky_2}{x} & \frac{ky_2z}{x} \\ -z_2 & y_2 & 0 \\ -1 & 0 & y_2 \end{bmatrix}. \tag{56b}$$

Unfortunately, these matrices have irrational functional factors. Using (11) we find the following equation for the scalar factors:

$$\frac{st_2}{ts_1} \doteq \frac{y_1}{y_2}. \tag{57}$$

One can easily verify that (57) is satisfied by

$$t = s = \frac{1}{y} \quad \text{and} \quad t = \frac{1}{y_1}, s = \frac{1}{y_2}, \tag{58}$$

which both yield rational Lax pairs. The factors t, s in (58) are related to those in (55). Using (31a), t in (55) can be written as

$$t = \sqrt[3]{\frac{x}{(p - k)y_1^2 y x_1}}. \tag{59}$$

After applying (22) with $a = \sqrt[3]{x/y}$, one can simplify the cube root to find $t = 1/y_1$, where the trivial factor $1/\sqrt[3]{p-k}$ has been canceled. A further application of (22) with $a = y$ then yields $t = 1/y$. The connections between the choices for s are similar.

An alternate form of a Lax pair is possible. Had the original constraint given by (40a) been expressed as

$$z_3 = x_3 y + z, \tag{60}$$

the substitutions would become

$$x_3 = \frac{\tilde{f}}{\tilde{F}}, \tag{61a}$$

$$y_3 = \frac{\tilde{g}}{\tilde{F}}, \tag{61b}$$

$$z_3 = \frac{\tilde{f}y + \tilde{F}z}{\tilde{F}}. \tag{61c}$$

With $\phi = \left[\begin{smallmatrix} \tilde{f} \\ \tilde{g} \\ \tilde{F} \end{smallmatrix}\right]$, $\mathcal{L}$ would then be given by

$$\mathcal{L} = t \begin{bmatrix} \frac{px_1y}{x} & -\frac{kyy_1}{x} & 0 \\ 0 & y & z - z_1 \\ -1 & 0 & y_1 \end{bmatrix}. \tag{62}$$

Note that the matrices (53) and (62) are gauge equivalent as defined in (12) with

$$\mathcal{G} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/y & -z/y \\ 0 & 0 & 1 \end{bmatrix}. \tag{63}$$

4 Implementation

4.1 Consistency Around the Cube

The CAC property has been used to identify integrable PΔEs [3, 10]. As shown in both examples, the information gained from the process of verifying CAC is also crucial to the computation of the corresponding Lax pair. In some sense the lattice equation is its own Lax pair, cf. the discussion in [14].

For scalar PΔEs, CAC is a simple concept that can be verified by hand or (interactively) with a computer algebra system (CAS) such as MATHEMATICA or MAPLE. Hereman [9] designed software to compute Lax pairs of scalar PΔEs defined on a quadrilateral. For systems of PΔEs with edge equations the verification of the CAC property can be tricky and the order in which substitutions are carried out is important. Designing a symbolic manipulation package that fully automates the steps is quite a challenge [7].

Naively, one could first generate the comprehensive system that represents the PΔEs on each face of the cube and then ask a CAS to solve it. To be consistent around the cube, that system should have a unique solution for $\mathbf{x}_{123}$. Wolf [30] discusses the computational challenges of verifying the CAC property for scalar PΔEs in three dimensions [26] due to the astronomical size of the overdetermined system that has to be solved. Even for PΔEs in two dimensions, in particular, those involving edge equations, automatically solving such a system often exceeds the capabilities of current symbolic software packages. It is therefore necessary to verify CAC in a more systematic way like one would do with pen on paper. Computer code [7] for automated verification of the CAC property carries out the following steps:

1. Solve the initial PΔE for $\mathbf{x}_{12}$. Solve the equations on the *bottom* and *left* faces for $\mathbf{x}_{13}$ and $\mathbf{x}_{23}$, respectively. Generate the equations for the *back*, *right* and *top* equations and solve each for $\mathbf{x}_{123}$. This produces three expressions for the components of $\mathbf{x}_{123}$.
2. Evaluate and simplify the solutions $\mathbf{x}_{123}$ using $\mathbf{x}_{12}, \mathbf{x}_{13}$, and $\mathbf{x}_{23}$. Use the constraints between the components of $\mathbf{x}, \mathbf{x}_1, \mathbf{x}_2$, and $\mathbf{x}_3$ arising from the edge equations to check consistency at every level of the computation.
3. Finally, verify if the three expressions for the components of $\mathbf{x}_{123}$ are indeed equal. If so, the system of PΔEs is consistent around the cube and one can proceed with the computation of the Lax matrices.

4.2 Computation of a Lax Pair

Assuming the given PΔE is CAC, the following steps are then taken to calculate a Lax pair:

1. Introduce fractional expressions (e.g., $\frac{f}{F}$, $\frac{g}{G}$, etc.) for the various components of $\mathbf{x}_3$ in order to linearize the numerators and denominators of the expressions for $\mathbf{x}_{13}$ in terms of f, F, g, G, etc.
2. Further simplify the components of $\mathbf{x}_3$ using the edge equations (if present in the given PΔE).

3. Substitute the simplified expressions for $\mathbf{x}_3$ into $\mathbf{x}_{13}$ and again examine if the numerators and denominators are linear in f, F, g, G, etc.
4. If $\mathbf{x}_{13}$ is not yet "linearized", reduce the degree of freedom (e.g., by setting $G = F$, etc.) and repeat this procedure until the numerators and denominators of the components of $\mathbf{x}_{13}$ are linear in f, F, g, etc.
5. Use the fractional linear expressions of $\mathbf{x}_{13}$ to generate the "core" Lax matrix L_c.
6. Use the determinant method (see (27)) to compute a possible scaling factor t.
7. The Lax matrix is then $L = tL_c$. The matrix $M = sM_c$ follows from L by replacing p by q and $\mathbf{x}_1$ by $\mathbf{x}_2$.

4.3 Verification of the Lax Pair

Finally, verify the Lax pair by substitution into the Lax equation (11). Unfortunately, the determinant method gives s and t in irrational form, introducing, e.g., square or cubic roots into the symbolic computations. In general, symbolic software is limited in simplification of expressions involving radicals. The impact of the presence of radical expressions can be reduced by careful simplification. Notice that (19) can be written as

$$\frac{(st_2)}{(ts_1)}(L_c)_2 M_c - (M_c)_1 L_c \doteq 0. \tag{64a}$$

Bringing all common factors from the matrix products up front gives

$$\left(\frac{st_2}{ts_1}\frac{\mathrm{CF}_{L_2M}}{\mathrm{CF}_{M_1L}}\right)\tilde{L}_2\tilde{M} - \tilde{M}_1\tilde{L} \doteq 0, \tag{64b}$$

where CF_X stands for a common factor of all the entries of a matrix X. Hence, $\mathrm{CF}_{L_2M}\tilde{L}_2\tilde{M} = (L_c)_2 M_c$ and $\mathrm{CF}_{M_1L}\tilde{M}_1\tilde{L} = (M_c)_1 L_c$. The computed Lax pair is correct if

$$\left(\frac{st_2}{ts_1}\frac{\mathrm{CF}_{L_2M}}{\mathrm{CF}_{M_1L}}\right) \doteq \pm 1 \tag{65a}$$

and thus

$$\pm\tilde{L}_2\tilde{M} - \tilde{M}_1\tilde{L} \doteq 0. \tag{65b}$$

To illustrate the verification procedure, consider Example 2 with t and s in (55). Here,

$$\frac{st_2}{ts_1} = \frac{\sqrt[3]{\frac{x}{(k-q)y_2^2(z-z_2)}}\sqrt[3]{\frac{x^2(y_2-y_1)^3(z-z_2)}{(k-p)yy_2(py_2(z_1-z)+qy_1(z-z_2))^2(z_1-z_2)}}}{\sqrt[3]{\frac{x}{(k-p)y_1^2(z-z_1)}}\sqrt[3]{\frac{x^2(y_2-y_1)^3(z-z_1)}{(k-q)yy_1(py_2(z_1-z)+qy_1(z-z_2))^2(z_1-z_2)}}}, \tag{66a}$$

$$\mathrm{CF}_{L_2M} = \frac{y_2}{x(y_1-y_2)} \quad \text{and} \quad \mathrm{CF}_{M_1L} = \frac{y_1}{x(y_1-y_2)}. \tag{66b}$$

The matrix $\tilde{L_2}\tilde{M}$ (which equals $\tilde{M_1}\tilde{L}$) is

$$\begin{bmatrix} -pqy(z_1-z_2) & ky(qy_1-py_2) & ky(py_2z_1-qy_1z_2) \\ pz_2(z-z_1)+qz_1(z_2-z) & \begin{array}{l} k(y_1z_2-y_2z_1) \\ \quad +py_2(z_1-z)+qy_1(z-z_2) \end{array} & kz(y_2z_1-y_1z_2) \\ p(z-z_1)+q(z_2-z) & k(y_1-y_2) & \begin{array}{l} kz(y_2-y_1)+py_2(z_1-z) \\ \quad +qy_1(z-z_2) \end{array} \end{bmatrix}. \tag{67}$$

Note that

$$\frac{\mathrm{CF}_{L_2M}}{\mathrm{CF}_{M_1L}} = \frac{y_2}{y_1}. \tag{68}$$

After multiplying (68) with (66a), the resulting expression can be simplified[1] into 1. Thus, both (65a) and (65b) are satisfied for the plus sign.

5 Results

The algorithm discussed in this paper is being implemented in MATHEMATICA and preliminary versions of the software [7, 9] are being verified against many known PΔEs. The Lax matrices L, including those for Examples 1 and 2 in the paper, are presented in Tables 1 through 5. The matrix M follows from the matrix L by the replacements $\mathbf{x}_1 \to \mathbf{x}_2$ and $p \to q$.

5.1 Scalar PΔEs

The scalar PΔEs given in Tables 1 and 2 are referenced by the names given in the classification by Adler, Bobenko, and Suris [3]. Each of these PΔEs involves the scalar field variable x and its shifts. The substitution used in the computation of a Lax pair is

$$x_3 = \frac{f}{F}. \tag{69}$$

Thus, the linear equations have the form (10a)–(10b), in which

$$\psi = \begin{bmatrix} f \\ F \end{bmatrix}. \tag{70}$$

Scaling factors can be computed with the determinant method but they are often irrational. If for scalar PΔEs the ratio $\frac{st_2}{ts_1}$ can be factored, i.e.,

$$\frac{st_2}{ts_1} = \frac{\mathcal{P}(x, x_1; p, q)\mathcal{Q}(x, x_1; p, q)}{\mathcal{P}(x, x_2; q, p)\mathcal{Q}(x, x_2; q, p)}, \tag{71}$$

[1] Use the MATHEMATICA function POWEREXPAND or simply cube the expression.

Table 2 Lax pairs of scalar P∆Es

Name	Equation	Matrix L	Alternate t values	Ref.
Q_1	$p(x-x_2)(x_1-x_{12})-q(x-x_1)(x_2-x_{12})+\delta^2pq(p-q)=0$	$t\begin{bmatrix} kx-\eta x_1 & -p(xx_1-\delta^2k\eta) \\ p & -(kx_1-\eta x)\end{bmatrix}$ where $\eta=k-p$, with $t=\frac{1}{\sqrt{-(\delta p-(x-x_1))(\delta p+(x-x_1))}}$	$t=\frac{1}{\delta p\pm(x-x_1)}$	[3]
Q_2	$p(x-x_2)(x_1-x_{12})-q(x-x_1)(x_2-x_{12})+pq(p-q)(x+x_1+x_2+x_{12})-pq(p-q)\big(p^2-pq+q^2\big)=0$	$t\begin{bmatrix} \eta(kp-x_1)+kx & \ell_{12} \\ p & -\eta(kp-x)+kx_1\end{bmatrix}$ where $\ell_{12}=-p\big[k\eta\big(k\eta+p^2-x-x_1\big)+xx_1\big]$, and $\eta=k-p$, with $t=\frac{1}{\sqrt{(x-x_1)^2-2p^2(x+x_1)+p^4}}$		[3]
Q_3	$\big(q^2-p^2\big)(xx_{12}+x_1x_2)+q\big(p^2-1\big)(xx_1+x_2x_{12})-p\big(q^2-1\big)(xx_2+x_1x_{12})-\frac{\delta^2}{4pq}\big(p^2-q^2\big)\big(p^2-1\big)\big(q^2-1\big)=0$	$t\begin{bmatrix} -4kp(\sigma px+\tau x_1) & -\gamma(\delta^2\sigma\tau-4k^2pxx_1) \\ -4k^2p\gamma & 4kp(\sigma px_1+\tau x)\end{bmatrix}$ where $\gamma=p^2-1,\ \sigma=k^2-1$ and $\tau=p^2-k^2$, with $t=\frac{1}{\sqrt{4p(px-x_1)(px_1-x)-\delta^2\gamma^2}}$	when $\delta=0$, $t=\frac{1}{px-x_1}$, or $t=\frac{1}{px_1-x}$	[3]
$(\alpha,\beta)-$ lattice	$\big((p-\alpha)x-(p+\beta)x_1\big)\big((p-\beta)x_2-(p+\alpha)x_{12}\big)-\big((q-\alpha)x-(q+\beta)x_2\big)\big((q-\beta)x_1-(q+\alpha)x_{12}\big)=0$	$t\begin{bmatrix} (p-\alpha)(p-\beta)x-\tau x_1 & -(k-\alpha)(k-\beta)xx_1 \\ (k+\alpha)(k+\beta) & -((p+\alpha)(p+\beta)x_1-\tau x)\end{bmatrix}$ where $\tau=p^2-k^2$ with $t=\frac{1}{\sqrt{((\beta-p)x+(\alpha+p)x_1)((\alpha-p)x+(\beta+p)x_1)}}$	$t=\frac{1}{(\alpha-p)x+(\beta+p)x_1}$, or $t=\frac{1}{(\beta-p)x+(\alpha+p)x_1}$	[16, 25]

then potential candidates for the scaling factors are

$$\begin{aligned} t &= \frac{1}{\mathcal{P}(x, x_1; p, q)}, \quad s = \frac{1}{\mathcal{P}(x, x_2; q, p)} \quad \text{and} \\ t &= \frac{1}{\mathcal{Q}(x, x_1; p, q)}, \quad s = \frac{1}{\mathcal{Q}(x, x_2; q, p)}. \end{aligned} \tag{72}$$

To verify that the candidate scaling factors actually work, $L = tL_c$ and $M = sM_c$ must satisfy (11). If they do work, such t and s are rational and preferred over the irrational scaling factors computed by the determinant method. The alternative rational scaling factors, obtained in this way, are listed for Q_1 and the (α, β)-equation in Table 2. The Lax pair for the (α, β)-equation was first presented in [25].

A similar situation happens with Q_3 when $\delta = 0$ where in addition to the irrational expression of t one has two rational alternatives, namely, $t = 1/(px - x_1)$ and $t = 1/(px_1 - x)$ which both satisfy

$$\frac{st_2}{ts_1} \doteq \frac{(q^2 - 1)(px - x_1)(px_1 - x)}{(p^2 - 1)(qx - x_2)(qx_2 - x)}. \tag{73}$$

For the equations A_1 and A_2 in Table 1, the ratio $\frac{st_2}{ts_1}$ is also of the form (71) but the choices (72) are not valid. The irrational forms of t and s as listed in Table 1 have to be used.

The Lax pair for Example 1, i.e., (4), follows from the one for H_3 by setting $\delta = 0$. However, when $\delta = 0$, the factors t and s can be taken rational (see (23) and (24)).

Further alternate rational factors are obtained using (22) for the Schwarzian, modified, Toda-modified Boussinesq equations as well as the Hietarinta systems.

5.2 Systems of PΔEs

5.2.1 Boussinesq Systems

For the Boussinesq system [15] in Table 3, $\psi = \begin{bmatrix} F \\ f \\ g \end{bmatrix}$. Substitution of

$$x_3 = \frac{f}{F}, \quad y_3 = \frac{g}{F}, \quad \text{and} \quad z_3 = \frac{fx - Fy}{F} \tag{74}$$

yields the Lax matrix given in Table 3.

Representing the edge constraint as $x_3 = \frac{z_3 + y}{x}$ requires

$$x_3 = \frac{\tilde{f} + \tilde{F}y}{\tilde{F}x}, \quad y_3 = \frac{\tilde{g}}{\tilde{F}}, \quad \text{and} \quad z_3 = \frac{\tilde{f}}{\tilde{F}}. \tag{75}$$

For $\phi = \begin{bmatrix} \tilde{F} \\ \tilde{f} \\ \tilde{g} \end{bmatrix}$, a resulting gauge equivalent $\mathcal{L}$ matrix is then

$$\mathcal{L} = \frac{1}{x} \begin{bmatrix} xx_1 - y & -1 & 0 \\ yy_1 & y_1 & -xx_1 \\ x(k - p + xy_1) - z(xx_1 - y) & z & -x^2 \end{bmatrix}, \tag{76}$$

Table 3 Lax pairs of systems of PΔEs

Name	Equation	Matrix L	Alternate t values	Ref.
Boussinesq	$z_1 - xx_1 + y = 0$ $z_2 - xx_2 + y = 0$ $(x_2 - x_1)(z - xx_{12} + y_{12}) - p + q = 0$	$t\begin{bmatrix} -x_1 & 1 & 0 \\ -y_1 & 0 & 1 \\ p - k - xy_1 + x_1 z & -z & x \end{bmatrix}$ with $t = 1$		[15]
Schwarzian Boussinesq	$z_1 - yx_1 - z = 0$ $z_2 - yx_2 - z = 0$ $xy_{12}(y_1 - y_2) - y(px_1 y_2 - qx_2 y_1) = 0$	$t\begin{bmatrix} \frac{pyx_1}{x} & -\frac{ky_1}{x} & \frac{kzy_1}{x} \\ -z_1 & y_1 & 0 \\ -1 & 0 & y_1 \end{bmatrix}$ with $t = \sqrt[3]{\frac{x}{y_1^2(z_1 - z)}}$	$t = \frac{1}{y}$, or $t = \frac{1}{y_1}$	[13]
Modified Boussinesq	$x_{12}(py_1 - qy_2) - y(px_2 - qx_1) = 0$ $xy_{12}(py_1 - qy_2) - y(px_1 y_2 - qx_2 y_1) = 0$	$t\begin{bmatrix} py_1 & 0 & -k \\ -kx_1 y & py & 0 \\ 0 & -\frac{kyy_1}{x} & \frac{px_1 y}{x} \end{bmatrix}$ with $t = \sqrt[3]{\frac{x}{x_1 y^2 y_1}}$	$t = \frac{1}{y}$, or $t = \frac{1}{y_1}$	[32]
Toda-modified Boussinesq	$y_{12}(p - q + x_2 - x_1) - (p-1)y_2 + (q-1)y_1 = 0$ $y_1 y_2(p - q - z_2 + z - 1) - (p-1)yy_2 + (q-1)yy_1 = 0$ $y(p + q - z - x_{12})(p - q + x - 2 - x_1) - (p^2 + p + 1)y_1 +$ $(q^2 + q + 1)y_2 = 0$	$t\begin{bmatrix} k + p - z & \frac{1+k+k^2}{y} & \ell_{13} \\ 0 & p - 1 & (1-k)y_1 \\ 1 & 0 & p - k - x_1 \end{bmatrix}$ where $\ell_{13} = (p^2 - k^2) - x_1(p + k) + z(k - p + x_1) -$ $\frac{y_1}{y}(p^2 + p + 1)$ with $t = \sqrt[3]{\frac{y}{y_1}}$	$t = 1$	[15]

where the gauge matrix, cf. (12), is given by

$$\mathcal{G} = \begin{bmatrix} 1 & 0 & 0 \\ y/x & 1/x & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{77}$$

5.2.2 Hietarinta Systems

For each system given in Table 4, $\psi = \left[\begin{smallmatrix} f \\ g \\ G \end{smallmatrix}\right]$. However, the substitutions are impacted by the edge equations in the systems. For system A-2, the edge constraint was represented as $x_3 = \frac{x+y_3}{z}$ resulting in substitutions of

$$x_3 = \frac{g + Gx}{Gz}, \quad y_3 = \frac{g}{G}, \quad \text{and} \quad z_3 = \frac{f}{G}. \tag{78}$$

Writing the edge constraint as $y_3 = x + x_3 z$ requires one to work with

$$x_3 = \frac{\tilde{g}}{\tilde{G}}, \quad y_3 = \frac{\tilde{G}x - \tilde{g}z}{\tilde{G}}, \quad \text{and} \quad z_3 = \frac{\tilde{f}}{\tilde{G}}. \tag{79}$$

Setting $\phi = \left[\begin{smallmatrix} \tilde{f} \\ \tilde{g} \\ \tilde{G} \end{smallmatrix}\right]$, the resulting gauge equivalent $\mathcal{L}$ matrix is given by

$$\mathcal{L} = \begin{bmatrix} \frac{y}{x} & \frac{k}{x} & -\frac{px_1 + yz_1}{x} \\ 0 & 1 & -x_1 \\ 1 & 0 & -z_1 \end{bmatrix}, \tag{80}$$

where L and $\mathcal{L}$ are connected as shown in (12) with

$$\mathcal{G} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/z & x/z \\ 0 & 0 & 1 \end{bmatrix}. \tag{81}$$

For system B-2, the edge constraint was represented as $x_3 = \frac{z+y_3}{x}$, resulting in

$$x_3 = \frac{g + Gz}{Gx}, \quad y_3 = \frac{g}{G}, \quad \text{and} \quad z_3 = \frac{f}{G}. \tag{82}$$

Representing the edge constraint as $y_3 = z + x_3 x$ yields

$$x_3 = \frac{\tilde{g}}{\tilde{G}}, \quad y_3 = \frac{\tilde{g}x - \tilde{G}z}{\tilde{G}}, \quad \text{and} \quad z_3 = \frac{\tilde{f}}{\tilde{G}}. \tag{83}$$

With $\phi = \left[\begin{smallmatrix} \tilde{f} \\ \tilde{g} \\ \tilde{G} \end{smallmatrix}\right]$ the resulting gauge equivalent $\mathcal{L}$ matrix is given by

$$\mathcal{L} = \begin{bmatrix} \delta + x & -(x\delta + y) & k - p + x_1(x\delta + y) - z_1(\delta + x) \\ 1 & 0 & -z_1 \\ 0 & 1 & -x_1 \end{bmatrix}, \tag{84}$$

Table 4 Lax pairs of systems of P∆Es

Name	Equation	Matrix L	Alternate t values	Ref.
A-2	$x_1z - y_1 - x = 0$ $x_2z - y_2 - x = 0$ $z_{12} - \frac{y}{x} - \frac{1}{x}\left(\frac{px_1 - qx_2}{z_1 - z_2}\right) = 0$	$t\begin{bmatrix} \frac{yz}{x} & \frac{k}{x} & \frac{kx - px_1z - yzz_1}{x} \\ -x_1z & z_1 & xz_1 \\ z & 0 & -zz_1 \end{bmatrix}$ with $t = \sqrt[3]{\frac{x}{x_1z^2z_1}}$	$t = \frac{1}{z}$, or $t = \frac{1}{z_1}$	[10]
B-2	$x_1x - y_1 - z = 0$ $x_2x - y_2 - z = 0$ $z_{12} + y - \delta(x_{12} - x) - xx_{12} - \frac{p-q}{x_1 - x_2} = 0$	$t\begin{bmatrix} -(\delta x + x^2) & \delta x + y & \ell_{13} \\ -xx_2 & z_2 & zz_2 \\ 0 & -1 & xx_2 - z \end{bmatrix}$ where $\ell_{13} = (z - xx_2)(\delta x + y) + z_2\left(\delta x + x^2\right) + x(q - k)$ with $t = \frac{1}{\sqrt[3]{x^2x_1}}$	$t = \frac{1}{x}$, or $t = \frac{1}{x_1}$	[10]
C-3	$y_1z - x_1 + x = 0$ $y_2z - x_2 + x = 0$ $z_{12} - \frac{\delta_2x + \delta_1}{y} - \frac{z}{y}\left(\frac{py_1z_2 - qy_2z_1}{z_1 - z_2}\right) = 0$	$t\begin{bmatrix} \frac{\delta_1 + \delta_2x - py_1z}{y} & \frac{kzz_1}{y} & -\frac{\delta_1z_1 + \delta_2xz_1}{y} \\ 0 & -z & x_1 - x \\ 1 & 0 & -z_1 \end{bmatrix}$ with $t = \sqrt[3]{\frac{y}{y_1z^2z_1}}$	$t = \frac{1}{z}$, or $t = \frac{1}{z_1}$	[10]
C-4	$y_1z - x_1 + x = 0$ $y_2z - x_2 + x = 0$ $z_{12} - \frac{xx_{12} - \delta_1}{y} - \frac{z}{y}\left(\frac{py_1z_2 - qy_2z_1}{z_1 - z_2}\right) = 0$	$t\begin{bmatrix} \frac{\delta_1 + xx_1 - py_1z}{y} & \frac{(k-x)zz_1}{y} & -\frac{(\delta_1 + x^2)z_1}{y} \\ 0 & -z & x_1 - x \\ 1 & 0 & -z_1 \end{bmatrix}$ with $t = \sqrt[3]{\frac{y}{y_1z^2z_1}}$	$t = \frac{1}{z}$, or $t = \frac{1}{z_1}$	[10]

where L and $\mathcal{L}$ are connected (cf. (12)) by

$$\mathcal{G} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/x & z/x \\ 0 & 0 & 1 \end{bmatrix}. \tag{85}$$

For system C-3, the edge constraint was represented as $x_3 = x + zy_3$ and

$$x_3 = \frac{Gx + gz}{Gx}, \quad y_3 = \frac{g}{G}, \quad \text{and} \quad z_3 = \frac{f}{G}. \tag{86}$$

Representing the edge constraint as $y_3 = \frac{x_3 - x}{z}$ requires

$$x_3 = \frac{\tilde{g}}{\tilde{G}}, \quad y_3 = \frac{\tilde{g} - \tilde{G}x}{\tilde{G}z}, \quad \text{and} \quad z_3 = \frac{\tilde{f}}{\tilde{G}}. \tag{87}$$

Letting $\phi = \left[\begin{smallmatrix} \tilde{f} \\ \tilde{g} \\ \tilde{G} \end{smallmatrix}\right]$, a gauge equivalent $\mathcal{L}$ matrix is

$$\mathcal{L} = \frac{1}{z} \begin{bmatrix} \frac{\delta_1 + x\delta_2 - pzy_1}{y} & \frac{kz_1}{y} & -\frac{(\delta_1 + x\delta_2 + kx)z_1}{y} \\ x_1 & -z_1 & 0 \\ 1 & 0 & -z_1 \end{bmatrix}, \tag{88}$$

with gauge matrix

$$\mathcal{G} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & z & x \\ 0 & 0 & 1 \end{bmatrix}. \tag{89}$$

For system C-4, the edge constraint was represented as $x_3 = x + zy_3$. Hence,

$$x_3 = \frac{Gx + gz}{Gx}, \quad y_3 = \frac{g}{G}, \quad \text{and} \quad z_3 = \frac{f}{G}. \tag{90}$$

Representing the edge constraint as $y_3 = \frac{x_3 - x}{z}$ requires

$$x_3 = \frac{\tilde{g}}{\tilde{G}}, \quad y_3 = \frac{\tilde{g} - \tilde{G}x}{\tilde{G}z}, \quad \text{and} \quad z_3 = \frac{\tilde{f}}{\tilde{G}}. \tag{91}$$

With $\phi = \left[\begin{smallmatrix} \tilde{f} \\ \tilde{g} \\ \tilde{G} \end{smallmatrix}\right]$, a resulting gauge equivalent $\mathcal{L}$ matrix is

$$\mathcal{L} = \frac{1}{z} \begin{bmatrix} \frac{\delta_1 + xx_1 - pzy_1}{y} & \frac{(k - x)z_1}{y} & -\frac{(\delta_1 + kx)z_1}{y} \\ x_1 & -z_1 & 0 \\ 1 & 0 & -z_1 \end{bmatrix}, \tag{92}$$

with gauge matrix

$$\mathcal{G} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & z & x \\ 0 & 0 & 1 \end{bmatrix}. \tag{93}$$

5.2.3 Two-Component pKdV and NLS Lattices

In finding a Lax pair for the two-component pKdV system [31] given in Table 5, the initial substitutions are

$$x_3 = \frac{f}{F} \quad \text{and} \quad y_3 = \frac{g}{G}, \tag{94}$$

which lead to the proper form of the components of $\mathbf{x}_{13}$. Thus, the resulting Lax pair comprises 4×4 matrices, as the linear equations involve the auxiliary vector

$$\psi = \begin{bmatrix} f \\ F \\ g \\ G \end{bmatrix}. \tag{95}$$

Also, an additional scaling factor is introduced by the disparate substitutions. In this case, the constraints on the scaling factors become

$$tT = \frac{1}{\sqrt{\det L_c}} = \frac{1}{p-k}. \tag{96}$$

Hence, one can take $t = T = 1$.

For the lattice NLS system [31] given in Table 5, one is only able to solve for x_{13} and x_{23} despite having equations referencing y. Thus, the substitution of $x_3 = \frac{f}{F}$ suffices to linearize the components of $\mathbf{x}_{13}$. The resulting Lax matrices, L and M, are 2×2 matrices and

$$\psi = \begin{bmatrix} f \\ F \end{bmatrix}. \tag{97}$$

6 Conclusion

We gave a detailed review of a three-step method [6, 14] to compute Lax pairs for scalar PΔEs defined on quadrilaterals and subsequently applied the method to systems of PΔEs. It was shown that for systems involving edge equations the derivation of Lax pairs can be quite tricky.

The paper also serves as a repository of Lax pairs, not only for the scalar integrable PΔEs classified by Adler, Bobenko, and Suris [3], but for systems of PΔEs including the discrete potential KdV equation, as well as various nonlinear Schrödinger and Boussinesq-type lattices. Previously unknown Lax pairs are presented for PΔEs recently derived by Hietarinta [10].

Preliminary software [9] is available to compute Lax pairs of scalar PΔEs defined on quadrilaterals. The extension of the code to systems of PΔEs is a nontrivial exercise. In the near future we hope to release a fully automated MATHEMATICA package [7] for the computation (and verification) of Lax pairs of two-dimensional PΔEs systems defined on quadrilaterals.

Table 5 Lax pairs of systems of PΔEs

Name	Equation	Matrix L	Alternate t values	Ref.
C-2.1	$x_{12} - \frac{x_2 z_1 - x_1 z_2}{z_1 - z_2} = 0$ $z_{12} + z x_{12} - \frac{z(p z_2 - q z_1)}{z_1 - z_2} = 0$	$t\begin{bmatrix} -z_1 & x_1 & 0 \\ z z_1 & -z(p + x_1) & k z z_1 \\ 0 & 1 & -z_1 \end{bmatrix}$ with $t = \frac{1}{\sqrt[3]{z z_1^2}}$	$t = \frac{1}{z}$, or $t = \frac{1}{z_1}$	[10]
C-2.2	$x_{12} - \frac{x_2 z_1 - x_1 z_2}{z_1 - z_2} = 0$ $z_{12} + \delta \frac{z}{x} - \frac{z}{x}\left(\frac{p x_1 z_2 - q x_2 z_1}{z_1 - z_2}\right) = 0$	$t\begin{bmatrix} -z_1 & x_1 & 0 \\ \frac{k z z_1}{x} & -\frac{z}{x}(\delta + p x_1) & \frac{\delta z z_1}{x} \\ 0 & 1 & -z_1 \end{bmatrix}$ with $t = \sqrt[3]{\frac{x}{z z_1^2 x_1}}$	$t = \frac{1}{z}$, or $t = \frac{1}{z_1}$	[10]
Two-component pKdV	$(x - x_{12})(y_1 - y_2) - p^2 + q^2 = 0$ $(y - y_{12})(x_1 - x_2) - p^2 + q^2 = 0$	$\begin{bmatrix} 0 & 0 & tx & t(p^2 - k^2 - x y_1) \\ 0 & 0 & t & -t y_1 \\ T y & T(p^2 - k^2 - x_1 y) & 0 & 0 \\ T & -T x_1 & 0 & 0 \end{bmatrix}$ with $t = T = 1$		[12, 31]
Lattice NLS	$y_1 - y_2 - y\big((x_1 - x_2)y + p - q\big) = 0$ $x_1 - x_2 + x_{12}\big((x_1 - x_2)y + p - q\big) = 0$	$t\begin{bmatrix} -1 & x_1 \\ y & k - p - y x_1 \end{bmatrix}$ with $t = 1$		[12, 31]

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