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FAST TRACK COMMUNICATION

Sufficient number of integrals for the pth-order Lyness equation

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Abstract
In this communication, we present a sufficient number of first integrals for the Lyness equation of arbitrary order. We first use the staircase method (Quispel *et al* 1991 *Physica* A **173** 243–66) to construct integrals of a derivative equation of the Lyness equation. Closed-form expressions for the integrals are given based on a non-commutative Vieta expansion. The integrals of the Lyness equation then follow directly from these integrals. Previously found integrals for the Lyness equation arise as special cases of our new set of integrals.

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1. Introduction

We consider the pth-order Lyness equation or the p-dimensional Lyness mapping

$$u_n u_{n+p} = a + u_{n+1} + u_{n+2} + \cdots + u_{n+p-1}. \tag{1}$$

This equation is a generalization of the equation which was first written by Lyness in 1942 [15] where $p = 2$ and $a = 1$. Recently, equation (1) and its particular cases have been studied by several authors [2, 3, 5–8, 13]. One important aspect in the study of the Lyness equation (1), or difference equations in general, is finding its integrals (invariants). Up to now, only three integrals of the pth-order Lyness equation have been found. The first invariant was given in [7, 11, 12]. The second and the third invariants were discovered in 2004 by Gao *et al* with the help of computer algebra [8], as well as fourth invariants for the seventh- and eighth-order Lyness equations. The authors conjectured that the pth-order Lyness equation has up to $\lfloor (p+1)/2 \rfloor$ integrals, which is a sufficient number of integrals for complete integrability in the sense of Liouville–Arnold (a $2N$-dimensional symplectic map has N functionally independent integrals which are in involution with respect to the symplectic structure [4, 25]).

In a recent paper [9], Grammaticos *et al* have pointed out that the Lyness equation (1) satisfies two criteria of integrability namely having singularity confinement and polynomial growth. This means that the Lyness equation is a good candidate for integrability in the sense of Liouville–Arnold. The authors of [9] also bilinearized the following consequence of equation (1), which is equation (1) minus its upshifted version:

$$u_{n+p}(1+u_n) = u_{n+1}(1+u_{n+p+1}). \tag{2}$$

We will call equation (2) the derivative Lyness equation. In [9], it was shown that the associated bilinear equation is a reduction of the integrable Hirota–Miwa equation. However, this does not allow us to calculate first integrals of the derivative equation.

There exists a method, called the staircase method, to construct integrals for a certain class of ordinary difference equations. This method was introduced in [16, 17] and generalised in [20, 23, 24]. Integrals were obtained for ordinary difference equations (OΔEs), derived as travelling wave reductions of integrable partial difference equations (PΔEs), i.e. PΔEs which exhibit a Lax pair. A so-called monodromy matrix is constructed by taking a product of Lax matrices (L, M) along a one-period segment of the periodic staircase. By expanding the trace of the monodromy matrix in powers of the spectral parameter, one will obtain integrals for the corresponding OΔE. To give closed-form expressions for integrals, in [22] we used this method in conjunction with a non-commutative Vieta expansion and a certain way of splitting Lax matrices. In that paper, integrals for reductions of ABS equations [1] were expressed in terms of multi-sums of products, Ψ.

In this communication, we observe that the derivative Lyness equation (2) is a travelling wave reduction of a certain two-dimensional integrable PΔE. It turns out therefore that one can use the staircase method to obtain integrals for the derivative Lyness equation. Moreover, we will show that the Lax matrix L for equation (2) fits in the framework of [22], i.e. we obtain the integrals of the derivative Lyness equation in closed form. Integrals of the pth-order Lyness equation (1) are then obtained directly from those of the derivative Lyness equation. This settles the conjecture given in [8] in the affirmative. Furthermore, this result indicates that the Lyness equation is a good candidate for complete integrability in the Liouville–Arnold sense.

The communication is organized as follows. In section 2, we will recall the staircase method for $(p, -1)$-travelling wave reductions. In section 3, we will give a Lax pair for the partial difference equation associated with equation (2). This Lax pair is derived from a Lax pair for equation (26) in [14] and a gauge transformation. Then, we will give closed-form expressions for integrals of the derivative Lyness equation (2). Once again, these integrals can be expressed in terms of multi-sums of products, Ψ, introduced in [22]. We will derive integrals of the original equation (1) from the integrals of the derivative Lyness equation in section 5. One will see that all three integrals given in [8] can be derived as special cases from our closed-form expressions. Finally, we give some relations between the integrals and we discuss the functional independence of the integrals.

2. Staircase method for $(p, -1)$-travelling wave reductions

In this section, we recall the staircase method which was introduced in [16, 17]. We consider a two-dimensional PΔE with field variable u,

$$f(u_{l,m}, u_{l+1,m}, u_{l,m+1}, u_{l+1,m+1}; \alpha) = 0, \tag{3}$$

and parameters $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$. An (s_1, s_2)-reduction can be performed to find solutions that satisfy the periodicity condition $u_{l,m} = u_{l+s_1,m+s_2}$ [20, 23, 24]. For our purpose here, we take $s_1 = p$ and $s_2 = -1$. We introduce a similarity variable $n = l + mp$. Then, $u_{l,m}$ satisfies the periodicity $u_{l,m} = u_{l+p,m-1}$, and the PΔE reduces to the OΔE

$$f_n = f(u_n, u_{n+1}, u_{n+p}, u_{n+p+1}; \alpha) = 0. \tag{4}$$

We suppose that equation (3) arises as the compatibility condition of two linear equations, that is it has a Lax pair. A Lax pair $L_{l,m}$, $M_{l,m}$ for a PΔE (3) is a pair of matrices that satisfies, cf [18],

$$L_{l,m}M_{l,m}^{-1} - M_{l+1,m}^{-1}L_{l,m+1} = 0, \tag{5}$$

for solutions of the equation. Similarly, an OΔE has a Lax pair if there are non-singular matrices $\mathcal{L}_n$, $\mathcal{M}_n$ that satisfy

$$\mathcal{M}_n\mathcal{L}_n - \mathcal{L}_{n+1}\mathcal{M}_n = 0. \tag{6}$$

The monodromy matrix $\mathcal{L}_n$ for the $(p, -1)$-reduction is given by, cf [17],

$$\mathcal{L}_n = M_n^{-1} \overset{\curvearrowleft}{\prod_{i=0}^{p-1}} L_{i+n}, \tag{7}$$

where the inversely ordered product is

$$\overset{\curvearrowleft}{\prod_{i=a}^{b}} L_i := L_b L_{b-1} \ldots L_{a+1} L_a. \tag{8}$$

Taking $\mathcal{M}_n = L_n$, we obtain a Lax pair $\mathcal{L}_n$, $\mathcal{M}_n$ for the reduced OΔE (4) from the Lax pair of the corresponding PΔE (3). This was first observed in [18]. Lax pairs for OΔEs obtained from general (s_1, s_2)-reductions have been studied in [20].

It follows from equation (6) that the trace of $\mathcal{L}_n$ is invariant under the map obtained from an OΔE. Since the Lax matrices generally depend on a spectral parameter, integrals for the OΔE (4) are obtained by expanding the trace of the monodromy matrix (or powers there of, or its determinant) in powers of the spectral parameter.

3. Lax pair for the derivative Lyness equation

The derivative Lyness equation (2) is a $(p, -1)$-travelling reduction of the following PΔE:

$$u_{l,m+1}(1 + u_{l,m}) = u_{l+1,m}(1 + u_{l+1,m+1}). \tag{9}$$

This equation is known as the 'discrete Lotka–Voltera equation of type I' [10], and is a discrete form of KdV [19]. A Lax pair for a generalisation of this equation has been given in [14].

It is easy to see that equation (9) is equivalent to the equation

$$\left(u_{l,m} - \tfrac{1}{2}\right)\left(u_{l+1,m} + \tfrac{1}{2}\right) = \left(u_{l+1,m+1} - \tfrac{1}{2}\right)\left(u_{l,m+1} + \tfrac{1}{2}\right), \tag{10}$$

using the transformation $u_{l,m} \mapsto -u_{m,l} - 1/2$ and interchanging l and m, which equals equation (31) in [14], taking $\alpha_j = 1/2$. Using the Lax pair given in [14] for equation (10), we find the following Lax pair for equation (9):

$$L_{l,m} = \begin{pmatrix} \lambda & -u_{l+1,m} \\ \dfrac{1}{u_{l+1,m}+1} & \dfrac{\lambda u_{l+1,m}}{u_{l+1,m}+1} \end{pmatrix}, \qquad M_{l,m} = \begin{pmatrix} \lambda & -(u_{l,m}+1)u_{l,m+1} \\ 1 & 0 \end{pmatrix}. \tag{11}$$

Now we will use a gauge transformation to obtain a more suitable form of a Lax pair for this equation. Recall that if a PΔE has a Lax pair $(L_{l,m}, M_{l,m})$, then a gauge matrix $G_{l,m}$, which can depend on dependent/independent variables and on the spectral parameter, will give us a new Lax pair $(\widetilde{L}_{l,m}, \widetilde{M}_{l,m})$ where

$$\widetilde{L}_{l,m} = G_{l+1,m}L_{l,m}G_{l,m}^{-1}, \qquad \widetilde{M}_{l,m} = G_{l,m+1}M_{l,m}G_{l,m}^{-1}. \tag{12}$$

Applying the gauge

$$G_{l,m} = \begin{pmatrix} 1/\lambda & 0 \\ 0 & 1 \end{pmatrix} \tag{13}$$

to the Lax pair $(L_{l,m}/\lambda, M_{l,m}/\lambda)$, we obtain a new Lax pair for equation (9) as follows:

$$\widetilde{L}_{l,m} = \begin{pmatrix} 1 & \dfrac{-u_{l+1,m}}{\lambda^2} \\ \dfrac{1}{u_{l+1,m}+1} & \dfrac{u_{l+1,m}}{u_{l+1,m}+1} \end{pmatrix}, \qquad \widetilde{M}^{-1}_{l,m} = \begin{pmatrix} 0 & 1 \\ \dfrac{-\lambda^2}{u_{l,m+1}(u_{l,m}+1)} & \dfrac{\lambda^2}{u_{l,m+1}(u_{l,m}+1)} \end{pmatrix}. \tag{14}$$

We will show in the next section that this new Lax pair does fit in the framework of [22].

4. Closed-form expressions for integrals of the derivative Lyness equation

In this section, we will give closed-form expressions for integrals of the derivative Lyness equation (2). These integrals again can be expressed in terms of multi-sums of products, Ψ [22]. We also calculate the integrating factors of these integrals. Recall that $\Lambda_n(u_n, u_{n+1}, \ldots, u_{n+p+1})$ is an integrating factor corresponding to an integral $I_n = I(u_n, u_{n+1}, \ldots, u_{n+p})$ of equation (4), $f_n = 0$, if

$$I_{n+1} - I_n = f_n \Lambda_n. \tag{15}$$

Although invariance of the integrals is implied by the staircase method, we will prove it directly by using the properties of the multi-sums of products, Ψ, and at the same time, we obtain the integrating factors. From now on, for our convenience, we write u_i instead of u_{n+i}.

We denote $k = -(\lambda^2+1)/\lambda^2$, and we split the reduced $\widetilde{L}$ matrices as we did in [22]. We have

$$\widetilde{L}_i = k u_{i+1} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & u_{i+1} \\ \dfrac{1}{u_{i+1}+1} & \dfrac{u_{i+1}}{u_{i+1}+1} \end{pmatrix} = r_i\left(kH + s_i A^i_i\right), \tag{16}$$

where

$$r_i := u_{i+1}, \qquad s_i := \frac{1}{u_{i+1}(u_{i+1}+1)}, \qquad H := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \text{and} \qquad A^i_j := \begin{pmatrix} a_i & a_i b_j \\ 1 & b_j \end{pmatrix},$$

with $a_i := u_{i+1} + 1$ and $b_i := u_{i+1}$. Therefore, applying lemma 8 in [22], we obtain the following:

$$\overset{\curvearrowleft}{\prod_{i=a}^{b}} \widetilde{L}_i = \left(\sum_{r=0}^{b-a+1} X^{a,b}_r k^r \right) \prod_{i=a}^{b} r_i,$$

with

$$X^{a,b}_r = \Psi^{a+1,b-2}_{r-1} H A^{b-1}_a + \Psi^{a+2,b-1}_{r-1} A^b_{a+1} H + \Psi^{a+2,b-2}_{r-2} H + \Psi^{a+1,b-1}_r A^b_a, \tag{17}$$

where $c_i := s_i(a_{i-1} + b_i)$ and

$$\Psi^{a,b}_r := s_{a-1} \left(\sum_{a \leqslant i_1, i_1+1 < i_2, i_2+1 < \cdots < i_{r-1}, i_{r-1}+1 < i_r \leqslant b} \prod_{j=1}^{r} f_{i_j} \right) \prod_{i=a}^{b+1} c_i, \tag{18}$$

with $f_i := s_{i+1}/(c_i c_{i+1})$.

By writing

$$(1+k)\widetilde{M}_0^{-1} = kH + \begin{pmatrix} 0 & 1 \\ \dfrac{1}{(u_0+1)u_p} & \dfrac{-1}{(u_0+1)u_p} \end{pmatrix},$$

we obtain

$$(k+1)\,\mathrm{Tr}(\widetilde{\mathcal{L}}_0) = \sum_{r=0}^{\lfloor (p+1)/2 \rfloor} k^r I_r^p, \tag{19}$$

where I_r^p is given as follows:

$$I_r^p = \left(\frac{u_1 \Psi_{r-1}^{1,p-3}}{(u_0 1)u_p} + \frac{\Psi_{r-1}^{2,p-2}}{u_0+1} + \frac{\Psi_{r-2}^{2,p-3}}{u_p(u_0+1)} + \frac{(u_0+u_1+1)\Psi_r^{1,p-2}}{u_0+1} + \Psi_{r-1}^{1,p-2} \right) \prod_{i=1}^{p} u_i, \tag{20}$$

with

$$s_i = \frac{1}{u_{i+1}(u_{i+1}+1)}, \qquad c_i = \frac{u_i+u_{i+1}+1}{u_{i+1}(u_{i+1}+1)}, \qquad f_i = \frac{u_{i+1}(u_{i+1}+1)}{(u_i+u_{i+1}+1)(u_{i+1}+u_{i+2}+1)}.$$

We also get

$$\frac{(k+1)}{(-k)^{p+1}} \mathrm{Det}\widetilde{\mathcal{L}}_0 = \frac{u_1 u_2 \ldots u_{p-1}}{(u_0+1)(u_1+1)\ldots(u_p+1)} =: I_G, \tag{21}$$

which gives us the inverse of the integral (4.1) given in [9]. Thus, we have the following theorem.

Theorem 1. *Let $p > 1$ and $0 \leqslant 2r \leqslant p+1$. We have the following:*

(i) I_r^p given by (20) is an integral of equation (2) with the integrating factor

$$\Lambda_r^p = \left(\frac{\Psi_{r-1}^{2,p-2}}{(u_0+1)(u_{p+1}+1)} + \frac{\Psi_{r-2}^{2,p-3}}{u_p(u_0+1)(u_{p+1}+1)} + \frac{u_1 u_{p+1} \Psi_r^{1,p-1}}{(u_0+1)u_p} - \frac{u_1 \Psi_r^{1,p-2}}{(u_0+1)u_p} \right) \prod_{i=2}^{p} u_i. \tag{22}$$

(ii) I_G given by (21) is also an integral of equation (2) with the integrating factor

$$\Lambda_{I_G} = \frac{u_2 u_3 \ldots u_{p-1}}{(u_0+1)(u_1+1)\ldots(u_p+1)(u_{p+1}+1)}. \tag{23}$$

We prove this theorem directly using the following properties of Ψ given in [22]:

$$\Psi_r^{n,m} = s_{m+1}(a_m + b_{m+1})\Psi_r^{n,m-1} + s_{m+1}\Psi_{r-1}^{n,m-2}, \tag{24}$$

$$\Psi_r^{n,m} = s_{n-1}(a_{n-1} + b_n)\Psi_r^{n+1,m} + s_{n-1}\Psi_{r-1}^{n+2,m}. \tag{25}$$

Proof.

(i) We write

$$S\left(I_r^p\right) - I_r^p = \prod_{i=2}^{p} u_i (A+B), \tag{26}$$

where S is a shift operator, i.e. $S(u_i) = u_{i+1}$ and

$$A = \left(\frac{u_2\Psi_{r-1}^{2,p-2}}{u_1+1} + \frac{\Psi_{r-2}^{3,p-2}}{u_1+1} + u_{p+1}\Psi_{r-1}^{2,p-1}\right) - u_1\left(\frac{\Psi_{r-1}^{2,p-2}}{u_0+1} + \frac{\Psi_{r-2}^{2,p-3}}{u_p(u_0+1)} + \Psi_{r-1}^{1,p-2}\right),$$

$$B = u_{p+1}\left(\frac{\Psi_{r-1}^{3,p-1}}{u_1+1} + \frac{(u_1+u_2+1)\Psi_r^{2,p-1}}{u_1+1}\right) - u_1\left(\frac{u_1\Psi_{r-1}^{1,p-3}}{(u_0+1)u_p} + \frac{(u_0+u_1+1)\Psi_r^{1,p-2}}{u_0+1}\right).$$

Using properties (24) and (25), we have

$$\begin{aligned}A &= \left(\frac{u_2\Psi_{r-1}^{2,p-2}}{(u_1+1)} + \frac{\Psi_{r-2}^{3,p-2}}{(u_1+1)} + \frac{(u_p+u_{p+1}+1)\Psi_{r-1}^{2,p-2}}{u_{p+1}+1} + \frac{\Psi_{r-2}^{2,p-3}}{u_{p+1}+1}\right)\\ &\quad - u_1\left(\frac{\Psi_{r-1}^{2,p-2}}{u_0+1} + \frac{\Psi_{r-2}^{2,p-3}}{u_p(u_0+1)} + \frac{(u_1+u_2+1)\Psi_{r-1}^{2,p-2}}{u_1(u_1+1)} + \frac{\Psi_{r-2}^{3,p-2}}{u_1(u_1+1)}\right)\\ &= (u_p(u_0+1) - u_1(u_{p+1}+1))\left(\frac{\Psi_{r-1}^{2,p-2}}{(u_0+1)(u_{p+1}+1)} + \frac{\Psi_{r-2}^{2,p-3}}{u_p(u_0+1)(u_{p+1}+1)}\right).\end{aligned}$$

Similarly, we get

$$B = (u_p(u_0+1) - u_1(u_{p+1}+1))\left(\frac{u_1u_{p+1}\Psi_r^{1,p-1}}{(u_0+1)u_p} - \frac{u_1\Psi_r^{1,p-2}}{(u_0+1)u_p}\right).$$

Therefore, we obtain an integrating factor as given in (22).

(ii) By direct calculation, one obtains the integrating factor Λ_{I_G} given by (23). □

5. Integrals of the *p*th-order Lyness equation

In this section, we construct integrals of the original Lyness equation from the corresponding derivative Lyness equation. We also show that the integrals given by Gao *et al* in [8] can be derived as a special case from the integrals obtained from our closed-form expressions.

Let us assume that $I(u_0, u_1, \ldots, u_p)$ is an integral of the derivative Lyness equation (2). This means that

$$I(u_0, u_1, \ldots, u_p) = I\left(u_1, u_2, \ldots, u_p, \frac{u_0u_p + u_p - u_1}{u_1}\right). \tag{27}$$

Now we define

$$Q(u_0, u_1, \ldots, u_{p-1}) = I\left(u_0, u_1, \ldots, u_{p-1}, \frac{a + u_1 + u_2 + \cdots + u_{p-1}}{u_0}\right). \tag{28}$$

Lemma 2. *$Q(u_0, u_1, \ldots, u_{p-1})$ is an integral of the Lyness equation (1), i.e.*

$$Q(u_0, u_1, \ldots, u_{p-1}) = Q\left(u_1, u_2, \ldots, u_{p-1}, \frac{a + u_1 + u_2 + \cdots + u_{p-1}}{u_0}\right). \tag{29}$$

Proof. It can be seen that

$$\begin{aligned}\text{LHS (29)} &= I\left(u_0, u_1, \ldots, u_{p-1}, \frac{a + u_1 + u_2 + \cdots + u_{p-1}}{u_0}\right)\\ &= I\left(u_1, u_2, \ldots, u_{p-1}, \frac{a + u_1 + u_2 + \cdots + u_{p-1}}{u_0}, \frac{u_0\frac{a+u_1+u_2+\cdots+u_{p-1}}{u_0} + \frac{a+u_1+u_2+\cdots+u_{p-1}}{u_0} - u_1}{u_1}\right)\end{aligned}$$

$$= I\left(u_1, u_2, \ldots, u_{p-1}, \frac{a+u_1+u_2+\cdots+u_{p-1}}{u_0}, \frac{a+u_2+\cdots u_{p-1}+\frac{a+u_1+u_2+\cdots+u_{p-1}}{u_0}}{u_1}\right)$$
$$= Q\left(u_1, u_2, \ldots, u_{p-1}, \frac{a+u_1+u_2+\cdots+u_{p-1}}{u_0}\right).$$

This means that $Q(u_0, u_1, \ldots, u_{p-1})$ is an integral of the Lyness equation. □

Using this theorem, it is easy to see that the first invariant in [8] is derived from I_G. For the others, by writing Ψ explicitly in terms of the variable u, one can show that the second and third invariants in [8] are equivalent to $I_0^p\big/I_G$ and $I^p_{\lfloor(p+1)/2\rfloor}\big/I_G$, respectively.

By using theorem 1 and lemma 2, we obtain closed-form expressions for integrals of the pth-order Lyness equation. In the following corollary of theorem 1, we have exploited property (24) to make the dependence of I_r^p on its last argument explicit.

Corollary 3. *We denote $T := a+u_0+u_1+\cdots+u_{p-1}$. Let $p > 2$ and $0 \leqslant 2r \leqslant p+1$. Then integrals of the pth-order Lyness equation are given by*

$$Q_{I_G} = \frac{u_0u_1\ldots u_{p-1}}{(u_0+1)(u_1+1)\ldots(u_{p-1}+1)T}, \tag{30}$$

$$\begin{aligned} Q_r^p = \Bigg(&\frac{u_1\Psi_{r-1}^{1,p-3}}{u_0+1} + \frac{(T+u_0u_{p-1})\Psi_{r-1}^{2,p-3}}{T(u_0+1)} + \frac{u_0\Psi_{r-2}^{2,p-4}}{T(u_0+1)} + \frac{\Psi_{r-2}^{2,p-3}}{u_0+1} + \frac{u_0(u_0+u_1+1)\Psi_{r-1}^{1,p-4}}{T(u_0+1)} \\ &+ \frac{(u_0+u_1+1)(T+u_0u_{p-1})\Psi_r^{1,p-3}}{T(u_0+1)} + \frac{(T+u_0u_{p-1})\Psi_{r-2}^{1,p-3}}{T} + \frac{u_0\Psi_{r-2}^{1,p-4}}{T}\Bigg)\prod_{i=1}^{p-1}u_i. \end{aligned} \tag{31}$$

6. Some relations between the integrals

In this section, we first give some relations between the integrals I_r^p of the derivative Lyness equation given by (20) and the integral I_G given by (21). This will lead to some relations between the integrals of the Lyness equation.

6.1. Some relations between the integrals of the derivative Lyness equation

We denote the monodromy matrix obtained by using the Lax pair $(L_{l,m}/\lambda, M_{l,m}/\lambda)$ by $\mathcal{L}_0$ where (L, M) are given by (11). We have

$$\mathcal{L}_0 = \left(\frac{M_0}{\lambda}\right)^{-1}\frac{L_{p-1}}{\lambda}\frac{L_{p-2}}{\lambda}\ldots\frac{L_0}{\lambda}.$$

It is easy to see that $\widetilde{\mathcal{L}}_0 = G_0^{-1}\mathcal{L}_0G_0$, where G_0 is given by (13). This implies that

$$\mathrm{Tr}(\widetilde{\mathcal{L}}_0) = \mathrm{Tr}\,\mathcal{L}_0. \tag{32}$$

Recall that integrals I_r^p are obtained by expanding $(k+1)\,\mathrm{Tr}\,\widetilde{\mathcal{L}}_0$ in powers of $k = -1-1/\lambda^2$, see equation (19). Now we expand $(k+1)\,\mathrm{Tr}\,\mathcal{L}_0$ in powers of $1/\lambda$. Equating the lowest coefficient and the coefficient for $1/\lambda^2$, we obtain the following relations.

Proposition 4. *Let $p \geqslant 2$, and let I_r^p and I_G be given in (20) and (21), respectively. Then we have*

$$\sum_{r=0}^{\lfloor(p+1)/2\rfloor} (-1)^r I_r^p = -I_G, \tag{33}$$

$$\sum_{r=1}^{\lfloor(p+1)/2\rfloor} (-1)^r r I_r^p + 1 = (u_0 u_p - u_1 - u_2 - \cdots - u_{p-1} - p) I_G. \tag{34}$$

A proof of this proposition will be given in the appendix.

Remark 5. Note that $u_0 u_p - u_1 - u_2 - \cdots - u_{p-1}$ is also an integral of the derivative Lyness equation with integrating factor 1.

6.2. Some relations between the integrals of the Lyness equation

Using proposition 4, we obtain the following linear relations between the integrals of the Lyness equation.

Corollary 6. *Let $p \geqslant 3$, and let Q_r^p and Q_{I_G} be given by (31) and (30), respectively. Then we have*

$$\sum_{r=0}^{\lfloor(p+1)/2\rfloor} (-1)^r Q_r^p = -Q_{I_G}, \tag{35}$$

$$\sum_{r=0}^{\lfloor(p+1)/2\rfloor} (-1)^r (p - a - r) Q_r^p = 1. \tag{36}$$

7. Conclusion

We have presented closed-form expressions for integrals of the pth-order Lyness equation and its derivative equation. Using Maple, for $p \leqslant 11$, for random values u_i, we have found that the set of integrals $\left\{I_0^p, I_1^p, \ldots, I_{\lfloor(p+1)/2\rfloor}^p\right\}$ of the derivative Lyness equation is functionally independent. Taking particular values for the variables, we are able to check orders that are a bit higher. We found functional independence for $p \leqslant 15$ when $u_0 = u_1 = \cdots = u_{p-1} = 1$ and $u_p = 2$. For the pth-order Lyness equation, the set $\left\{Q_0^p, Q_1^p, \ldots, Q_{\lfloor(p+1)/2\rfloor}^p\right\}$ is not functionally independent due to the linear relation (36). However, the rank of the Jacobian matrix seems to be $R = \lfloor(p+1)/2\rfloor$ for $p \leqslant 11$ and for the random value a and u_i. In particular, this is the case for $p \leqslant 15$ and for the random value of a when $u_0 = u_1 = \cdots = u_{p-1} = 1$. In fact, for all $p \leqslant 15$ the determinant of $2\frac{a+p}{a+p-2}$ times the upper-left $R \times R$ part of the Jacobian is

$$\pm \frac{3^{q_{p-1}}}{2^{g_{p+1}}} \frac{a - \lfloor p/2\rfloor}{a + p}, \tag{37}$$

where the powers of 3 are given by quarter squares $q_p = \lfloor p^2/4\rfloor$ and the powers of 2 are generalized pentagonal numbers, cf [21],

$$g_p = \left\lfloor \frac{p}{2} \right\rfloor \left(3\left\lfloor \frac{p}{2}\right\rfloor - (-1)^p\right) \Big/ 2.$$

We conjecture formula (37) to be true for any p. The existence of $\lfloor(p+1)/2\rfloor$ functionally independent integrals in the set our integrals for the Lyness equation would be enough for complete integrability in the sense of Liouville–Arnold.

Another problem is to see if the Lyness equation has a symplectic structure or not. One way to approach this problem is using a Lagrangian to derive the symplectic structures [4]. We have found a Lagrangian for a double copy of equation (2). However, to construct symplectic structures for the Lyness mapping itself is still an open problem.

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Appendix. Proof of proposition 4

We write

$$L_i/\lambda = \begin{pmatrix} 1 & 0 \\ 0 & \frac{u_{i+1}}{u_{i+1}+1} \end{pmatrix} + \frac{1}{\lambda}\begin{pmatrix} 0 & -u_{i+1} \\ \frac{1}{u_{i+1}+1} & 0 \end{pmatrix}, \tag{A.1}$$

and

$$(k+1)(M_0/\lambda)^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{-1}{(u_0+1)up} \end{pmatrix} + \frac{1}{\lambda}\begin{pmatrix} 0 & -1 \\ \frac{1}{(u_0+1)u_p} & 0 \end{pmatrix}. \tag{A.2}$$

Equalling the lowest coefficient of the following:

$$(k+1)\,\mathrm{Tr}(\widetilde{\mathcal{L}}_0) = (k+1)\mathrm{Tr}\mathcal{L}_0,$$

we obtain the first relation (33). Now equalling the coefficient of $1/\lambda^2$, we obtain

$$\sum_{r=1}^{\lfloor (p+1)/2 \rfloor} (-1)^r r I_r^p = \left(\sum_{1\leqslant r<s\leqslant p} \prod_{i=1}^{r} u_i \prod_{i=r}^{s-1}(1+u_i) \prod_{i=s+1}^{p} u_i - \sum_{r=1}^{p}\left(u_p \prod_{i=r+1}^{p} u_i \prod_{i=0}^{r-1}(1+u_i) \right.\right.$$
$$\left.\left. + \prod_{i=1}^{r} u_i \prod_{i=r}^{p}(1+u_i)\right)\right) \frac{1}{u_p(u_0+1)(u_1+1)\dots(u_p+1)}. \tag{A.3}$$

In this proof, we use the following identity with $m < n$:

$$\prod_{i=m}^{n}(1+u_i) - \sum_{m<s\leqslant n} \prod_{i=m}^{s-1}(1+u_i) \prod_{i=s+1}^{n} u_i = (1+u_m) \prod_{i=m+1}^{n} u_i. \tag{A.4}$$

This identity is proved by expanding its left-hand side. We have

$$\begin{aligned}
\text{LHS (A.4)} &= \left(\prod_{i=m}^{n}(1+u_i) - \prod_{i=m}^{n-1}(1+u_i) \right) - \sum_{m<s\leqslant n-1} \prod_{i=m}^{s-1}(1+u_i) \prod_{i=s+1}^{n} u_i \\
&= \left(\prod_{i=m}^{n-1}(1+u_i) - \prod_{i=m}^{n-2}(1+u_i) \right) u_n - \sum_{m<s\leqslant n-2} \prod_{i=m}^{s-1}(1+u_i) \prod_{i=s+1}^{n} u_i \\
&\quad \vdots \\
&= (1+u_m) \prod_{i=m+1}^{n} u_i = \text{RHS (A.4)}.
\end{aligned}$$

We write

$$\sum_{r=1}^{\lfloor (p+1)/2\rfloor} (-1)^r r I_r + 1 = \frac{P_1}{P_2},$$

where

$$P_1 = u_p \prod_{i=0}^{p}(u_i+1) + \sum_{1\leqslant r<s\leqslant p} \prod_{i=1}^{r} u_i \prod_{i=r}^{s-1}(1+u_i) \prod_{i=s+1}^{p} u_i - \sum_{r=1}^{p} u_p \prod_{i=r+1}^{p} u_i \prod_{i=0}^{r-1}(1+u_i) + \prod_{i=1}^{r} u_i \prod_{i=r}^{p}(1+u_i),$$

$$P_2 = u_p \prod_{i=0}^{p}(u_i+1).$$

Thus, we only need to show that

$$P_1 = (u_0u_p - u_1 - u_2 - \cdots - u_{p-1} - p)\prod_{i=1}^{p} u_i.$$

We have

$$\begin{aligned} P_1 &= \left(u_p \prod_{i=0}^{p}(u_i+1) - \sum_{r=1}^{p} u_p \prod_{i=r+1}^{p} u_i \prod_{i=0}^{r-1}(1+u_i)\right) + \sum_{1\leqslant r<s\leqslant p-1} \prod_{i=1}^{r} u_i \prod_{i=r}^{s-1}(1+u_i) \prod_{i=s+1}^{p} u_i \\ &\quad + \sum_{1\leqslant r\leqslant p-1} \prod_{i=1}^{r} u_i \prod_{i=r}^{p-1}(1+u_i) - \sum_{r=1}^{p} \prod_{i=1}^{r} u_i \prod_{i=r}^{p}(1+u_i) \\ &= u_p(1+u_0)\prod_{i=1}^{p} u_i + \sum_{1\leqslant r<s\leqslant p-1} \prod_{i=1}^{r} u_i \prod_{i=r}^{s-1}(1+u_i) \prod_{i=s+1}^{p} u_i - \sum_{1\leqslant r\leqslant p-1} \prod_{i=1}^{r} u_i \prod_{i=r}^{p-1}(1+u_i) u_p \\ &\quad - (1+u_p)\prod_{i=1}^{p} u_i \\ &= (u_0u_p - 1)\prod_{i=1}^{p} u_i + \sum_{r=1}^{p-1}\prod_{i=1}^{r} u_i \left(\sum_{r<s\leqslant p-1} \prod_{i=r}^{s-1}(1+u_i) \prod_{i=s+1}^{p-1} u_i - \prod_{i=r}^{p-1}(1+u_i)\right) u_p \\ &= (u_0u_p - 1)\prod_{i=1}^{p} u_i - \sum_{r=1}^{p-1}(1+u_r)\prod_{i=1}^{p} u_i = (u_0u_p - p - u_1 - u_2 - \cdots - u_{p-1})\prod_{i=1}^{p} u_i. \end{aligned}$$

This proves our statement.

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