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Somos-4 and Somos-5 are arithmetic divisibility sequences

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ABSTRACT

We provide an elementary proof to a conjecture by Robinson that multiples of (powers of) primes in the Somos-4 sequence are equally spaced. We also show, almost as a corollary, for the generalized Somos-4 sequence defined by $\tau_{n+2}\tau_{n-2} = \alpha\tau_{n+1}\tau_{n-1} + \beta\tau_n^2$ and initial values $\tau_1 = \tau_2 = \tau_3 = \tau_4 = 1$, that the polynomial $\tau_n(\alpha, \beta)$ is a divisor of $\tau_{n+k(2n-5)}(\alpha, \beta)$ for all $n, k \in \mathbb{Z}$ and establish a similar result for the generalized Somos-5 sequence. The proofs involve elliptic divisibility sequences, for which we also show that primes are equally spaced, and a nice property of subsets $S \subset \mathbb{Z}$ for which $t, s \in S \Rightarrow 2s - t \in S$.

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1. Introduction

An arithmetic divisibility sequence is neither a divisibility sequence, nor an arithmetic sequence. We define it to be a sequence $\{f_n\}_{n\in\mathbb{Z}}$ such that there is a function $d : \mathbb{Z} \to \mathbb{N}$, called the common difference function, such that

$$\forall n \in \mathbb{Z} : d(n) \mid (m - n) \implies f_n \mid f_m.$$

Clearly, if for some sequence $\{f_n\}_{n\in\mathbb{Z}}$ prime powers appear in arithmetic progressions that sequence is an arithmetic divisibility sequence. Furthermore, we then have the following nice property: for all $m = n + kd(n)$ and $r = m + sd(m)$, with $n, k, s \in \mathbb{Z}$, there is an $l \in \mathbb{Z}$ such that $r = n + ld(n)$. Primes do not have to appear in arithmetic progressions, e.g. consider the periodic arithmetic divisibility sequence

$$f_n = \begin{cases} 12 & \text{if } n \equiv 0 \bmod 6, \\ 1 & \text{if } n \equiv 1, 5 \bmod 6, \\ 6 & \text{if } n \equiv 2, 4 \bmod 6, \\ 4 & \text{if } n \equiv 3 \bmod 6. \end{cases}$$

where we have even made sure that $f_n \mid f_m \implies d(n) \mid (m - n)$. The above mentioned nice property also holds if the common difference function is polynomial. This follows

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from the fact that $p(x)$ divides $p(x + p(x)q(x))$ for all polynomials $p(x), q(x)$, which can be seen by substituting $y = x + p(x)q(x)$ in $(y - x) \mid (p(y) - p(x))$.

Sequences $\{\tau_n\}$ defined by

$$\tau_{n+k-2}\tau_{n-2} = \alpha\tau_{n+k-3}\tau_{n-1} + \beta\tau_{n+k-4}\tau_n \tag{1}$$

possess the Laurent property, that is, all their terms are Laurent polynomials in their initial values. Equation (1) arises as a special case of the Hirota–Miwa equation, whose integrability condition is equivalent to Laurentness [20]. For $k = 4$ and $k = 5$ these equations are called Somos-4 and Somos-5. The Laurent property implies in particular that if the initial values are 1, and α, β are integers, both recurrences define integer sequences. Considering α and β to be variables, we obtain polynomial sequences. The Laurent property was introduced by Hickerson, cf. [23], to prove the integrality of the next sequence, Somos-6, which does not have the form (1) but is a special case of the discrete BKP equation [20,25]. Initially the integrality, and secondly the Laurent property itself, seemed to be curious properties [10]. By now there is extended literature on equations that possess the Laurent property, have Laurentness, or display the Laurent phenomenon, [1,8,9,11,13,15,16,19,20], and the property is well understood in terms of cluster algebras [7], which have had a profound impact in diverse areas of mathematics, cf. [6].

The Somos-4 & 5 sequences also satisfy a strong Laurent property, e.g. for Somos-4 the terms are Laurent in τ_1, polynomial in $\tau_2, \ldots, \tau_k$, and polynomial in $\mathcal{I}$, an additional (invariant) rational function of the initial values [15]. Therefore, integral sequences may be obtained for other than unit initial values. The Somos-4 sequence with $\alpha = -1, \beta = 2$ and initial values $\tau_1 = \tau_2 = 1,\ \tau_3 = 2,\ \tau_4 = 3$ is an integer sequence which extends the Fibonacci numbers; we have $\tau_n = (-1)^{n+1}\tau_{-n}$ and $\tau_n = \tau_{n-1} + \tau_{n-2}$.

Few results on divisibility properties of *the* Somos-4 & 5 sequences (with $\alpha = \beta = 1$ and $\tau_1 = \cdots = \tau_k = 1$) are known. In [4] it is shown that every term beyond the fourth of the Somos-4 sequence has a primitive divisor, i.e. a prime which does not divide any preceding term. Jones and Rouse [17] show that the density of primes dividing at least one term of the Somos-4 sequence is 11/21. In [18] the authors establish that the terms of Somos-4 (with $\alpha = \beta = 1$) are irreducible Laurent polynomials in their initial values and pairwise co-prime. Robinson [23] showed that, for $k = 4, 5$, the ith and jth terms of the Somos-k sequence are relatively prime whenever $|i - j| \leq k$. He infers that both sequences are periodic modulo m for every $m \in \mathbb{N}$. In [23, section 4] he presents results about Somos-4 and Somos-5 obtained by calculation, for which he did not have general proofs. One observation he made is that if a prime p divides any term of Somos-k, then the multiples of p are equally spaced. In proof he added that Clifford S. Gardner proved this result, but a reference was not provided. This conjecture and generalizations have been proven by C.S. Swart in her thesis [26]. She also proved, for more general Somos-4 sequences, that either all multiples of a prime p are divisible by exactly the same power of p, or $\exists r \in \mathbb{N}$ such that $\forall m \geq r$ some term is divisible by p^m exactly, cf. Section 8. The following problem is still open: to prove that for particular Somos sequences for all but finitely many primes p the multiples of p are not divisible by exactly the same power of p.

In this short paper we provide a simple proof, in Section 6, that for all primes $p \in \mathbb{P}$ the set $\{n \in \mathbb{Z} : p^m \mid \tau_n\}$ has either less than 2 elements, or its elements form a complete arithmetic sequence. This also holds for rational Somos-4 & 5 sequences, for primes that do not appear in the denominator of the invariant function, as long as the initial values are pairwise co-prime. Moreover, it holds true in elliptic divisibility sequences (EDS), see Section 3.

We employ an elementary result from number theory, which we haven't found in the literature, namely that the terms of a subset $S \subset \mathbb{Z}$ with the property that $2s - t \in S$ for all $s, t \in S$, form an arithmetic sequence. A proof of this result is presented in Section 2.

Combining the result on the appearance of primes with a symmetry of the equation and taking unit initial values we find Somos-k sequences, with $k = 4, 5$ which are arithmetic divisibility sequences with common difference function $d_k(n) = 2n - k - 1$, i.e. we prove the polynomial divisibility

$$\tau_n(\alpha, \beta) \mid \tau_{n+ld_k(n)}(\alpha, \beta) \quad \forall n, l \in \mathbb{Z}.$$

We briefly investigate different initial values which give rise to Laurent sequences with the same divisibility property, as well as initial values that give rise to polynomial sequences with slightly different divisibility properties. We note that if one could prove for odd primes p that the multiples of p in *the* Somos-4 & 5 sequences are not divisible by exactly the same power of p then we would have the following.

Conjecture 1: *Let τ_n be the terms of the Somos-k sequence with $k = 4, 5$, $\alpha = \beta = 1$, $\tau_1 = \cdots = \tau_k = 1$, $d = 2n - k - 1$ and $q = \tau_n$ when $(k + 1) \nmid n$ or $q = \tau_n/2$ when $(k + 1) \mid n$. Then*

$$q^{m+1} \mid \tau_l \Leftrightarrow l = n + \left(\frac{q^m - 1}{2} + kq^m\right) d.$$

A similar result for the extended Fibonacci sequence would provide an alternative proof for the fact that the $(m + 1)$-st power of the nth Fibonnaci number divides the (nf_n^m)th Fibonnaci number, as was shown by M. Cavachi [3], at the age of 16. We note that the case $n = 3$ is exceptional; here $2^{m+2} \mid f_{2^m 3}$.

2. Sets of differences

In [2] a *set of differences* is defined to be a subset S of $\mathbb{N}$ such that $s, t \in S$, $s > t \Rightarrow s-t \in S$. A useful result is that the elements in sets of differences are multiples of the smallest element. Extending the definition to subsets of $\mathbb{Z}$ a similar statement will be used to provide a proof that EDS are divisibility sequences and that powers of primes are equally spaced, in the next section.

For our purposes we define a subset S of $\mathbb{Z}$ to be a *modified set of differences* if $s, t \in S \Rightarrow 2s - t \in S$.

Lemma 2: *A modified set of differences has less than two elements, or its terms form a complete arithmetic sequence.*

Proof: Let s, t be elements of a modified set of differences V. We first show, by induction, that

$$(k + 1)s - kt \in V, \text{ and } (k + 1)t - ks \in V, \tag{2}$$

for all $k \in \mathbb{Z}$. By definition it is true for $k = 1$. Assuming the statement we obtain that $2s - ((k + 1)s - kt) = kt - (k - 1)s$ and $2t - ((k + 1)s - kt) = (k + 2)t - (k + 1)s$ are in V, as well as the same expressions with s, t interchanged.

If $s = t$ all the above elements coincide, the set may only have one element. If s and t are distinct we may take $s < t$ in which case $(k+1)t - ks$ with $k \in \mathbb{N}$ provides plenty of positive elements. Let $a, b > a$ be the smallest and the next smallest positive elements of V and denote $d = b - a$. As $a - d = 2a - b \in V$ and a is the smallest positive element we have $a < d$. Taking $s = a$ and $t = 2a - b$ in the first element of (2) we find that $kd + a \in V$ for all $k \in \mathbb{Z}$. We show the converse, $s \in V \Rightarrow d \mid (s - a)$.

For any $s \in V$ we may write $s = qd + r$ with $0 \le r < d$. If $r = a$ then we are done. Let $t = qd + a \in V$. Assume $r < a$. The element $z = (k+1)s - kt = s - k(a - r)$, with $k = \lfloor \frac{s}{a-r} \rfloor$, is in V and $0 \le z < a - r$ which is impossible, as a is the smallest positive element of V. Assuming $r > a$, we have, with $k = \lfloor \frac{t}{r-a} \rfloor$, that $p = (k+1)t - ks = t - k(r-a)$ is in V and $0 \le p < r - a$. We also have, replacing k with $k - 1$, that $q = p + (r - a) \in V$. As $p < d < b$ we need $p = a$. But with $p = a$ we have $a < q = r < d < b$ which contradicts that b is the second smallest positive element. □

3. EDS: integrality and divisibility

EDS were introduced by Morgan Ward [27,28] as sequences of integers $\{a_n\}_{n=0}^{\infty}$, that satisfy, for all $m \ge n \ge 1$,

$$a_{m+n}a_{m-n} = \begin{vmatrix} a_n a_{m-1} & a_{n-1}a_m \\ a_{n+1}a_m & a_n a_{m+1} \end{vmatrix} \tag{3}$$

and $n \mid m \Rightarrow a_n \mid a_m$. He calls a sequence proper if $a_0 = 0, a_1 = 1, a_2^2 + a_3^2 \neq 0$, and shows that a proper solution to (3) is an EDS if and only if a_2, a_3, a_4 are integers and $a_2 \mid a_4$. Ward first shows by induction, that all terms are integers, and then by another induction step, the divisibility property.

Ward's proof of integrality is generalized by Hone and Swart [15], who proved the following strong Laurent property for Somos-4: the terms τ_n are polynomials in $\alpha, \beta, \tau_1^{\pm 1}, \tau_2, \tau_3, \tau_4$, and $\mathcal{I}$, where

$$\mathcal{I} = \alpha^2 + \beta T, \quad T = \frac{\tau_1^2\tau_4^2 + \alpha(\tau_2^3\tau_4 + \tau_1\tau_3^3) + \beta\tau_2^2\tau_3^2}{\tau_1\tau_2\tau_3\tau_4} \tag{4}$$

Therefore, if $\tau_1 = \pm 1$ and $\mathcal{I}$ is an integer, the sequence consists of integers. We remark that taking $n = 2$ in (3) gives us a special Somos-4 sequence,

$$a_{m+2}a_{m-2} = a_2^2 a_{m+1}a_{m-1} - a_1 a_3 a_m^2. \tag{5}$$

Taking $\alpha = a_2^2, \beta = -a_1a_3, \{\tau_i = a_i\}_{i=1}^4$ and $a_1^2 = 1$ in (4) we find $\mathcal{I} = -a_4/a_2$, whose integrality implies that the sequence $\{a_m\}$ consists of integers.

For Somos-5 we have the following strong Laurent property [15, Theorem 3.7]: The terms $\tau_{n>0}$ of Somos-5 are polynomial in $\alpha, \beta, \tau_1^{\pm 1}, \tau_2^{\pm 1}, \tau_3, \tau_4, \tau_5$, and $\mathcal{J}$, where

$$\mathcal{J} = \beta + \alpha S, \quad S = \frac{(\tau_1\tau_5 + \alpha\tau_3^2)(\tau_1\tau_4^2 + \tau_2^2\tau_5) + \beta\tau_2\tau_3^3\tau_4}{\tau_1\tau_2\tau_3\tau_4\tau_5}.$$

The sequence is an integer sequence if $\tau_1, \tau_2 \in \{\pm 1\}$, $\mathcal{J}$ is an integer and one can find three consecutive integers preceding two units.

An EDS uniquely extends to a sequence over $\mathbb{Z}$. Taking $m = 2$ in (5) one finds $a_0 = 0$. Taking $m = 1$ in (5) we find $a_3 a_{-1} = -a_1 a_3 a_1^2 \Rightarrow a_{-1} = -1$ (assuming $a_3 \neq 0$). Taking $m = -1$ and $n = k - 1$ in (3) gives $a_{k-2} a_{-k} = -a_{k-2} a_k a_{-1}^2 \Rightarrow a_{-k} = -a_k$ (assuming $a_{k-2} \neq 0$). For $k = 2$ this doesn't work, but there exist a determining equation, e.g. take $m = 0$ in (5). One way to deal with the occurrence of zeros other than a_0 is to take the initial values as parameters (let a_4 be a multiple of a_2), generate a polynomial sequence, and then specialize. See [15, Appendix A] for another discussion on zeros.

The divisibility property follows by showing that $V_k = \{m \in \mathbb{Z} : a_k \mid a_m\}$, with $k > 1$, is a set of differences with at least two elements. To show this we employ a second family of recurrences

$$a_1 a_2 a_{m+n+1} a_{m-n} = \begin{vmatrix} a_n a_{m-1} & a_{n-1} a_m \\ a_{n+2} a_{m+1} & a_{n+1} a_{m+2} \end{vmatrix}. \tag{6}$$

First, by taking the initial values as parameters (with $a_4 = a_2 \bar{a}_4$) or assuming subsequent initial values to be co-prime, using (5) one can inductively show that $\gcd(a_m, a_{m+1}){=}1$ for all $m > 0$.

Secondly, we show that $g = \gcd(a_m, a_{m+2})$ is a divisor of a_{2k} for all $k \in \mathbb{Z}$. From (5) it follows that $g \mid a_2^2$, and hence $(g, a_3) = 1$. Equation (6) with $(n, m) = (m - 1, 2)$, taking $a_1 = 1$ and dividing by a_2 yields

$$a_{m+2} a_{m-3} = a_3 a_{m-2} a_{m+1} - \bar{a}_4 a_{m-1} a_m$$

which implies $g \mid a_{m-2}$. The same equation also implies that $\gcd(a_m, a_{m-2}) \mid a_{m+2}$. We obtain $g = a_2$, and $a_2 \mid a_{2k}$ and $a_2 \nmid a_{2k-1}$ for all $k \in \mathbb{Z}$. In particular, m is even.

Next, taking $m = s$ and $n = t - s$ in (3) gives

$$a_t a_{2s-t} = a_{t-s}^2 a_{s-1} a_{s+1} - a_{t-s-1} a_{t-s+1} a_s^2.$$

If $s, t \in V_k$ then $\gcd(a_k, a_{s\pm1}) = 1$, and so $a_k \mid a_{t-s}^2$. Taking $m = s$ and $n = t - s - 1$ in (6) gives

$$a_2 a_t a_{2s-t+1} = a_{t-s-1} a_{t-s} a_{s-1} a_{s+2} - a_{t-s-2} a_{t-s+1} a_s a_{s+1}. \tag{7}$$

If $(a_k, a_{s+2}) = 1$ then $a_k \mid a_{t-s-1} a_{t-s}$. Together with a_{t-s-1} and a_{t-s} being co-prime we find that $s, t \in V_k \Rightarrow t - s \in V_k$. Suppose $g = (a_k, a_{s+2}) \neq 1$. Since $a_k \mid a_s$, s, t, and k are even and $g = a_2$. We may divide (7) by a_2,

$$a_t a_{2s-t+1} = a_{t-s-1} a_{t-s} a_{s-1} \frac{a_{s+2}}{a_2} - \frac{a_{t-s-2}}{a_2} a_{t-s+1} a_s a_{s+1}.$$

As a_s is co-prime with $\frac{a_{s+2}}{a_2}$ we arrive at the same conclusion as before: $s, t \in V_k \Rightarrow t - s \in V_k$.

Because both 0 and k are elements of V_k we have $V_k = k\mathbb{Z}$. Thus, the polynomial sequence $\{a_n\}$ is an arithmetic divisibility sequence with common difference function

$$d(n) = \begin{cases} n & n \neq 0, \\ \infty & n = 0. \end{cases}$$

For special initial values the value of d at 0 can be finite.

Ward does not give much detail about the consistency of the family of recurrences (3). He does however provide an explicit solution for a_n in terms of the Weierstrass sigma function, and both (3) and (6) are direct consequences of the corresponding three-term relation, cf. [14]. For an algebraic approach we refer the reader to [21].

The fact that an EDS $\{a_n\}$ is a divisibility sequence does not imply that multiples of powers of primes are equally spaced. However, for an EDS whose initial values satisfy $\gcd(a_n, a_{n+1})=1$ an argument similar to the above shows that $V = \{n \in \mathbb{Z} : p^k \mid a_n\}$, with $p \in \mathbb{P}$, $k \in \mathbb{N}$ is a modified set of differences with at least two elements, and this implies the following theorem.

Theorem 3: *For an EDS in which subsequent (initial) values are co-prime the multiples of powers of primes are equally spaced.*

4. Companion EDS

Taking unit initial values for Somos-4,

$$\tau_1 = \tau_2 = \tau_3 = \tau_4 = 1, \tag{8}$$

we have $T = 1 + 2\alpha + \beta$, and $\mathcal{I} = \alpha^2 + \beta T = (\alpha + \beta)^2 + \beta$. We define [15, Definition 1] an EDS by (5) and the initial values

$$a_2 = -\sqrt{\alpha},\ a_3 = -\beta,\ a_4 = \sqrt{\alpha}\mathcal{I}, \tag{9}$$

so $\{a_n\}$ satisfies the same recurrence as $\{\tau_n\}$. The sequence $\{a_n\}$ is the companion EDS for Somos-4, that is, the following families of recurrences are satisfied [14, Corollaries 1.2, 1.3],

$$\tau_{m+n}\tau_{m-n} = \begin{vmatrix} a_n\tau_{m-1} & a_{n-1}\tau_m \\ a_{n+1}\tau_m & a_n\tau_{m+1} \end{vmatrix}, \tag{10}$$

and

$$a_1 a_2 \tau_{m+n+1}\tau_{m-n} = \begin{vmatrix} a_n\tau_{m-1} & a_{n-1}\tau_m \\ a_{n+2}\tau_{m+1} & a_{n+1}\tau_{m+2} \end{vmatrix}. \tag{11}$$

Proofs of these facts can be found in [14,21], cf. [15].

To describe the companion EDS for Somos-5 with initial values

$$\tau_1 = \tau_2 = \tau_3 = \tau_4 = \tau_5 = 1, \tag{12}$$

we introduce an alternating sequence of functions

$$h_l = \begin{cases} 2\alpha + \beta & l \equiv 0 \bmod 2, \\ \alpha + 1 & l \equiv 1 \bmod 2, \end{cases}$$

whose product equals the above mentioned invariant $\mathcal{J} = h_l h_{l+1}$. Companion EDS are then given by

$$a_1^\epsilon = 1,\ a_2^\epsilon = \sqrt{h_\epsilon},\ a_3^\epsilon = \alpha,\ a_4^\epsilon = -\beta a_2,$$

and

$$a_{k+2}^\epsilon a_{k-2}^\epsilon = h_{k+\epsilon} a_{k+1}^\epsilon a_{k-1}^\epsilon - \alpha (a_k^\epsilon)^2.$$

We note that $a^{\epsilon}_{2m+1} = a^{\epsilon+1}_{2m+1}$ and $a^{\epsilon}_{2m}/\sqrt{h_\epsilon} = a^{\epsilon+1}_{2m}/\sqrt{h_{\epsilon+1}}$. The terms of the Somos-5 sequence satisfy the same families of recurrences (10,11) with $a_\bullet$ replaced by $a^{m+n}_\bullet$, see also [14, Corollary 2.12] and [21, Section 6]. For Somos-5 with general initial values the companion EDS can be defined by

$$a_1 = 1, a_2 = -\mu, a_3 = \alpha, a_4 = \mu\beta,\ a_{k+2}a_{k-2} = \mu^2 a_{k+1}a_{k-1} - \alpha a_k^2,$$

with $\mu^4 = \beta + \alpha\mathcal{J}$, see [15, Proof of Theorem 3.7].

5. Relative primeness

Both α and β are not divisors of τ_n. This can be seen by taking $\alpha = 0$ or $\beta = 0$. The corresponding solutions, for Somos-4, are $\tau_n = \beta^{k_n}$ and $\tau_n = \alpha^{l_n}$, where k_n and l_n satisfy the linear recurrences

$$k_{n+2} = 2k_n - k_{n-2} + 1, \quad l_{n+2} = l_{n+1} + l_{n-1} - l_{n-2} + 1.$$

In particular, they do not vanish. If α and β have a divisor in common its multiplicity will grow as ([24, A249020]). Considering the terms τ_n as polynomials in α and β, we can adjust the argument of Bergman, cf. [10], to prove that any four consecutive terms of our polynomial Somos-4 sequence are pairwise co-prime. The initial terms (8) are co-prime. Assume that $\{\tau_{n+i}\}_{i=-2}^{1}$ are pairwise co-prime, and let p be an irreducible divisor of τ_{n+2} with positive degree. Obviously p does not divide α or β, and therefore $p \mid a_n$ if and only if $p \mid a_{n-1}$ or $p \mid a_{n+1}$. By hypothesis this does not happen. Similarly for Somos-5 with pairwise co-prime initial values any five consecutive terms are pairwise co-prime.

6. Primes and divisibility in Somos-4 and Somos-5

Theorem 4: *Multiples of powers of primes are equally spaced in Somos-k sequences when initial values $\tau_1, \ldots, \tau_k$ are pairwise coprime, for $k = 4, 5$.*

Proof: Let τ_n denote the terms of either Somos-4 or Somos-5. We show that the set $W_q = \{n \in \mathbb{Z} : q \mid \tau_n\}$ is a modified set of differences. Here τ_n and q are elements in the polynomial ring $\mathbb{Z}[\alpha, \beta, \tau_1^{\pm1}, \tau_2^{\pm1}, \tau_3^{\pm1}, \tau_4^{\pm1}]$, or $\mathbb{Z}[\alpha, \beta, \tau_1^{\pm1}, \tau_2^{\pm1}, \tau_3^{\pm1}, \tau_4^{\pm1}, \tau_5^{\pm1}]$, respectively, whereas the a_n are elements of $\mathbb{Z}[\sqrt{\alpha}, \beta, \tau_1^{\pm1}, \tau_2^{\pm1}, \tau_3^{\pm1}, \tau_4^{\pm1}]$, or $\mathbb{Z}[\alpha, \beta, \mu, \tau_1^{\pm1}, \tau_2^{\pm1}, \tau_3^{\pm1}, \tau_4^{\pm1}, \tau_5^{\pm1}]$. Note for Somos-5, with unit initial values, and EDS pair defined in terms of h_l, whenever q divides a_n^{ϵ} it also divides $a_n^{\epsilon+1}$, and we may omit the upper index. Of course, one might take $q = \tau_k$, or let $q = p^h$ where $p \in \mathbb{P}$ does not occur in a denominator. Taking $m = s$ and $n = t - s$ in (10) gives us

$$\tau_t\tau_{2s-t} = a_{t-s}^2\tau_{s-1}\tau_{s+1} - a_{t-s-1}a_{t-s+1}\tau_s^2.$$

If $s, t \in W_q$ then $\gcd(q, \tau_{s\pm1}) = 1$ and hence $q \mid a_{t-s}^2$. Taking $m = s$ and $n = t - s - 1$ in (11) yields $q \mid a_{t-s}a_{t-s-1}$, and hence $q \mid a_{t-s}$. Taking $m = s$ and $n = t - s$ in (11) gives us

$$a_1a_2\tau_{t+1}\tau_{2s-t} = a_{t-s}a_{t-s+1}\tau_{s-1}\tau_{s+2} - a_{t-s-1}a_{t-s+2}\tau_s\tau_{s+1}.$$

We may divide by a_2 since one of indices in each of the products $a_{t-s}a_{t-s+1}$ and $a_{t-s-1}a_{t-s+2}$ is even. Therefore, we obtain $2s - t \in W_q$ from $\gcd(q, a_1\tau_{t+1}) = 1$, which is what we wanted to show. By lemma 2 it follows that W_q has less than 2 elements or there exist $n, d \in \mathbb{N}$ such that $m \in W_q \Rightarrow d \mid m - n$. ☐

Theorem 5: *The Somos-k sequences, for $k = 4, 5$, with initial values $\tau_1 = \cdots = \tau_k = 1$ are arithmetic divisibility sequences with common difference function $d(n) = 2n - k - 1$.*

Proof: We now specialize to initial values $\tau_1 = \cdots = \tau_k = 1$. Due to the symmetry of the recurrences, $\tau_{n+l} \leftrightarrow \tau_{n-l}$ for all l, we obtain $\tau_{-n+(k+1)/2} = \tau_{n+(k+1)/2}$ for all n. In terms of $d = 2n - k - 1$ we have $\tau_{n-d} = \tau_n$. So W_{τ_n} has at least two elements. As the degree of τ_l, with $0 < l < n$, is smaller than the degree of τ_n, we have $\tau_n \nmid \tau_l$ and hence d is the common difference. ☐

7. Equivalent sequences

It is possible to choose other initial values with the same symmetry. For Somos-4, taking $\tau'_1 = -\tau'_2 = -\tau'_3 = \tau'_4 = 1$ one gets the polynomial sequence $\tau'_n(\alpha, \beta) = (-1)^{\lfloor n/2 \rfloor}\tau_n(-\alpha, \beta)$, where $\tau_n(\alpha, \beta)$ is the sequence obtained from $\tau_1 = \tau_2 = \tau_3 = \tau_4 = 1$. More generally, starting from $\tau'_1 = \tau'_4 = 1$ and $\tau'_2 = \tau'_3 = \gamma$ we find a sequence in $\mathbb{Z}[\alpha, \beta, \gamma^{\pm 1}]$,

$$\tau'_n(\alpha, \beta) = \frac{\tau_n(\gamma^3\alpha, \gamma^4\beta)}{\gamma^{(n-1)(n-4)/2}}. \tag{13}$$

If $\beta = \gamma^2$ the sequence is polynomial, and for $n > 4$, with $d = 2n - 5$,

$$\begin{cases} n \equiv 1 \bmod 3: & \tau'_n \mid \tau'_m \Leftrightarrow d \mid (m - n), \\ n \not\equiv 1 \bmod 3: & \tau'_n \mid \tau'_m \Leftrightarrow d \mid (m - n) \text{ and } \frac{m-n}{d} \not\equiv 1 \bmod 3. \end{cases}$$

Furthermore, starting from $\tau'_1 = \tau'_4 = \delta$ and $\tau'_2 = \tau'_3 = \gamma$ we find a sequence in $\mathbb{Z}[\alpha, \beta, \gamma^{\pm 1}, \delta^{\pm 1}]$,

$$\tau'_n(\alpha, \beta) = \frac{\delta^{(n-2)(n-3)/2}}{\gamma^{(n-1)(n-4)/2}}\tau_n\left(\left(\frac{\gamma}{\delta}\right)^3\alpha, \left(\frac{\gamma}{\delta}\right)^4\beta\right). \tag{14}$$

Both (13) and (14) have the same divisibility properties as $\tau_n(\alpha, \beta)$.

For Somos-5, starting from $\tau'_1, \ldots, \tau'_5 = a, b, c, b, a$ we have

$$\tau'_n(\alpha, \beta) = \frac{a^{A_n}b^{B_n}}{c^{C_n}}\tau_n\left(\left(\frac{c}{a}\right)^2\alpha, \left(\frac{c}{a}\right)^3\beta\right), \tag{15}$$

where

$$A_n = \frac{n^2}{4} - \frac{3n}{2} + \frac{17 - (-1)^n}{8}, \quad B_n = \frac{1 + (-1)^n}{2}, \quad C_n = \frac{n^2}{4} - \frac{3n}{2} + \frac{13 + 3(-1)^n}{8}.$$

We get a polynomial sequence from initial values $\tau_1 = \tau_5 = 1$, $\tau_2 = \tau_4 = b$, and $\tau_3 = \alpha$. Here we find the divisibility, with $d = 2n - 6$, $\tau_n \mid \tau_m \Leftrightarrow d \mid (m - n)$ for $n > 5$, $\tau_n \mid \tau_m \Leftrightarrow n \mid m$ for $n = 2, 3$.

We note the sequences τ and τ' in this section are equivalent sequences in the sense of [26, Sect. 6.3], and that the slightly different divisibility properties are due to initial values having a common factor.

8. Robinson's observations

We state here some observations made in [23] and their current status. Robinson observed that

(1) the multiples of primes are equally spaced [such a prime is called *regular*]
(2) the gap (the common difference) is never much larger than p
(3) if p occurs then p^2 occurs, and its gap is p times the gap of p
(4) if p^i is the smallest occurring power, the gap of p^{i+l} will be p^l times the gap of p^i

Much of this is now understood, but not all. Hone and Swart [12,15,26] (as well as Naom Elkies, David Speyer, and Nelson Stephens in unpublished work, cf. [22]) have shown that the terms τ_n of a Somos-4 sequence correspond to rational points $Q+[n]P$ on an associated elliptic curve E. Christine Swart, in her thesis [26], studies elliptic curves over $\mathbb{Z}_{p^r}$; due to an equivalence

$$\tau_n \equiv 0 \bmod p^r \Leftrightarrow Q + [n]P = \mathcal{O}_{p^r},$$

if p^r occurs, the gap of p^r equals the order N_r of the (non-singular) point P in $E(\mathbb{Z}_{p^r})$. Swart has proved that either all powers p^k are regular, or all multiples of p are divisible by exactly the same power of p [26, Thm 7.6.6]. She obtained the structure of the gap-function [26, Thm 7.6.7] and explained (and improved) Robinson's bound on the gap exploiting the Hasse bound which bounds a multiple of the order of P, i.e. the number of points on $E(\mathbb{F}_p)$ within $2\sqrt{p}$ of $p+1$ [26, Thm 7.6.5]. Armed with the above mentioned theorem, the gap can be calculated from the order of P on $E(\mathbb{F}_p)$.

We conclude with a couple of examples of interest, using curves E and points P as given in [15].

- Somos-4 with $\alpha = \beta = \tau_1 = \cdots = \tau_4 = 1$. The prime 2 divides τ_m if and only if 5 divides m. Higher powers of 2 do not appear (the sequence mod 4 is periodic with period 10 and does not contain 0). It seems that for all $p \neq 2$ such that $\exists n \in \mathbb{N} : p \mid \tau_n$, all powers p^k are regular, and $N_r = p^{r-1}N_1$, where N_1 is the order of the point (1,1) on the curve $y^2 = 4(x^3 - x) + 1$ modulo p. To prove conjecture 1 it suffices to show that all powers of odd primes occurring are regular, one does not need $w = 1$ for all p. Here w, and below v, are defined as in [26, Thm 7.6.7], in particular w is the largest integer such that $p^k \mid \tau_n \implies w \leq k$.
- Somos-4 with $\alpha = -1$, $\beta = 2$, $(\tau_1, \ldots, \tau_4) = (1, 1, 2, 3)$ (which extends the Fibonacci sequence). For $p = 2$ we have $w = 1$ and $v = 2$, for $p = 3$ we have $w = 1$ and $N_1 = 4$. It seems that all powers of occurring primes are regular, $N_r = p^{r-w}N_1$ where N_1 is the order of the point $(\frac{7}{12}, \sqrt{-1})$ on the singular curve

$$y^2 = \frac{(6x-5)(5+12x)^2}{216}$$

over the field $\mathbb{F}_p[\sqrt{-1}]$. Again, we have not found a value of w different than 1.

- Somos-4 with $\alpha = 4$, $\beta = 9$, $\tau_1 = \tau_4 = 1$, $\tau_2 = \tau_3 = 3$. We have $3^k \mid \tau_m \Leftrightarrow k = 1$, $3 \nmid m$. Taking $p = 5$ we find $N_{r\geq 3} = 5^{r-3} N_1$ where $N_1 = 7$ is the order of the point $(55750/243, 2)$ on the curve given by

$$y^2 = 4x^3 - \frac{12428112196}{19683}x + \frac{1385503884676628}{14348907}$$

over $\mathbb{F}_5$.

- Somos-4 with $\alpha = 2$, $\beta = 5$, $(\tau_1, \ldots, \tau_4) = (1, 3, 2, 5)$. We have $7^k \mid \tau_m \Leftrightarrow k = 2$, $N_1 \mid m$, where $N_1 = 10$ is the order of $P = (223081/21600, \sqrt{2})$ on

$$y^2 = 4x^3 - \frac{48492460561}{38880000}x + \frac{10678311547192441}{1259712000000}$$

over $\mathbb{F}_7$ or over $\mathbb{F}_7[\sqrt{2}]$.

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