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Poisson Brackets of Mappings Obtained as $(q, -p)$ Reductions of Lattice Equations

Dinh T. Tran[1*], Peter H. van der Kamp[2], and G. R. W. Quispel[2***]**

[1] *School of Mathematics and Statistics, University of New South Wales, Sydney NSW 2052, Australia*

[2] *Department of Mathematics and Statistics, La Trobe University, Bundoora VIC 3086, Australia*

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Abstract—In this paper, we present Poisson brackets of certain classes of mappings obtained as general periodic reductions of integrable lattice equations. The Poisson brackets are derived from a Lagrangian, using the so-called Ostrogradsky transformation. The $(q, -p)$ reductions are $(p+q)$-dimensional maps and explicit Poisson brackets for such reductions of the discrete KdV equation, the discrete Lotka–Volterra equation, and the discrete Liouville equation are included. Lax representations of these equations can be used to construct sufficiently many integrals for the reductions. As examples we show that the $(3, -2)$ reductions of the integrable partial difference equations are Liouville integrable in their own right.

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1. INTRODUCTION

In this paper we derive Poisson structures (and symplectic structures) for several families of discrete dynamical systems. In general, of course, the problem of finding a Poisson structure for a given mapping (if it has such a structure) is not algorithmic. However, suppose the system in question is the periodic reduction of a partial difference equation, e. g.,

$$u_{l+1,m+1} = F(u_{l,m}, u_{l+1,m}, u_{l,m+1})$$

with

$$u_{l+q,m-p} = u_{l,m},$$

where F is a given function, $\mathbb{R}^3 \to \mathbb{R}$, and u is the unknown function, $\mathbb{Z}^2 \to \mathbb{R}$. In the simplest cases, $p = q$ resp. $p = 1$, Poisson structures were first found by inspection, cf. [15], resp. [21]. Later on, for the case $p = -q$ (pre) symplectic structures for reductions of Adler–Bobenko–Suris (ABS) equations were derived using the existence of the so-called three-leg form [1, 18]. Using the three-leg forms of ABS equations, we can derive Lagrangians for these equations which help us to find symplectic structures for the case where $p = \pm q$. This method is called the Hamiltonian–Lagrangian approach and it was introduced in [5, 24]. For the case $p = q$, using the discrete Legendre transformation we can bring reduction of potential Korteweg–de Vries (pKdV) to canonical coordinates [7]. For the case $p \neq \pm q$, the discrete analogue of the so-called Ostrogradsky transformation will help us to bring the map to canonical coordinates [4, 5, 24]; therefore we can derive Poisson structures of the original maps. We note that, in the continuous context, this transformation has been used to construct Hamiltonians from higher derivative Lagrangians.

*E-mail: T.D.Tran@UNSW.edu.au

**E-mail: P.vanderKamp@LaTrobe.edu.au

***E-mail: R.Quispel@LaTrobe.edu.au

The Hamiltonian–Lagrangian approach was used in [11][1)] to find the Poisson structure of $p = 1$ reductions of the discrete KdV equation.

In this paper we use the Ostrogradsky transformation to find Poisson structures for periodic reductions of several partial difference equations, including the discrete KdV equation, the plus-KdV equation, the discrete Lotka–Volterra equation, and the discrete Liouville equation. All the partial difference equations we discuss (except for one) are integrable, i. e., they possess a Lax representation, and this Lax representation can be used to derive Lax representations for the reductions, in their turn yielding integrals of the reductions, cf. [15, 16, 19]. As examples, we explicitly show that $p = 2$, $q = 3$ reductions of the integrable partial difference equations are Liouville integrable in their own right, i. e., they possess a sufficient number of functionally independent integrals in involution with respect to the derived Poisson structures.

2. SETTING

In this section, we give some basic notions for the rest of the paper. In particular, we will present the so-called discrete Ostrogradsky transformation to obtain Poisson brackets from a Lagrangian.

2.1. Completely Integrable Map

Recall that a map between two Poisson manifolds is called Poisson if it preserves the Poisson brackets. In particular, a d-dimensional mapping

$$\begin{aligned} \varphi : \mathbb{R}^d &\to \mathbb{R}^d \\ (x_1, x_2, \dots, x_d) &\mapsto (x_1^{'}, x_2^{'}, \dots, x_d^{'}) \end{aligned} \tag{2.1}$$

is called a Poisson map if

$$\{x_i^{'}, x_j^{'}\} = \{x_i, x_j\} \mid_{\mathbf{x}=\mathbf{x}'}, \tag{2.2}$$

where $\{,\}$ denotes the Poisson bracket in $\mathbb{R}^d$ (the same bracket in the domain and the co-domain). One sufficient condition for the map (2.1) to be Poisson is the existence of a structure matrix, Ω, such that

$$d\varphi \Omega(x) d\varphi^T = \Omega(x^{'}), \tag{2.3}$$

where $d\varphi$ is the Jacobian of the map (2.1). By structure matrix we mean that Ω is a $d \times d$ skew-symmetric matrix that satisfies the Jacobi identity [14]

$$\sum_l \left(\Omega_{li} \frac{\partial}{\partial x_l} \Omega_{jk} + \Omega_{lj} \frac{\partial}{\partial x_l} \Omega_{ki} + \Omega_{lk} \frac{\partial}{\partial x_l} \Omega_{ij} \right) = 0, \ \forall i, j, k. \tag{2.4}$$

The Poisson bracket between 2 smooth functions f and g is defined through this structure by the following formula:

$$\{f, g\} = \nabla f . \Omega . (\nabla g)^T,$$

where ∇f is the gradient of f. In this case, $\Omega_{ij} = \{x_i, x_j\}$. We can see that the entries in the LHS and RHS of (2.3) are $\{x_i^{'}, x_j^{'}\}$ and $\{x_i, x_j\} \mid_{\mathbf{x}=\mathbf{x}'}$, respectively.

If the Poisson structure is nondegenerate, i. e., Ω has full rank, one can define a closed two-form

$$w = \sum_{i<j} w_{ij} dx_i \wedge dx_j$$

through the Poisson structure $W = \Omega^{-1}$ and vice versa[2)]. Then the Poisson map is called a symplectic map and preserves the closed two-form W. For the case where the closed two-form is preserved by the map (2.1), but is degenerate, we call the two-form presymplectic [6].

A function $I(\mathbf{x})$ is called an integral of φ if $I(\mathbf{x}) = I(\mathbf{x}^{'})$. A set of smooth functions $\{I_1, I_2, \dots, I_k\}$ on an open domain D is called functionally independent on D if its Jacobian matrix dI has full rank on a set dense in D.

[1)] Another approach to obtain Poisson structures used in [11] is based on cluster algebras.

[2)] In the literature the symbol J is sometimes used instead of W. We reserve J to denote the Jacobian.

Definition 1. *The map φ is completely integrable if it preserves a Poisson bracket of rank $2r \leqslant d$ and it has $r + d - 2r = d - r$ functionally independent integrals which are pairwise in involution.*

2.2. $(q, -p)$ Reductions of Lattice Equations

In this section, we introduce the $(q, -p)$-reduction $(\gcd(p, q) = 1, q, p > 0)$ to obtain ordinary difference equations from lattice equations.

Given a lattice equation (see Fig. 1)

$$Q(u_{l,m}, u_{l+1,m}, u_{l,m+1}, u_{l+1,m+1}) = 0,$$

the $(q, -p)$ reduction can be described as follows. We introduce the periodicity condition $u_{l,m} = u_{l+q,m-p}$ and travelling wave reduction $v_n = u_{l,m}$, where $n = lp + mq$. For $0 \leqslant n \leqslant p + q - 1$, since $\gcd(p, q) = 1$ one can find unique $(l, m), l \geqslant 0, m \leqslant 0$ such that $|m|$ is smallest and $n = lp + mq$. The lattice equation reduces to the following ordinary difference equation

$$Q(v_n, v_{n+p}, v_{n+q}, v_{n+p+q}) = 0. \tag{2.5}$$

This equation gives us a $(p + q)$-dimensional map

$$(v_n, v_{n+1}, \ldots, v_{p+q-1}) \mapsto (v_{n+1}, v_{n+2}, \ldots, v_{n+p+q}),$$

where we have assumed that one can solve v_{n+p+q} uniquely from Eq. (2.5). For example, the $(3, -2)$-reduction is depicted in Fig. 1.

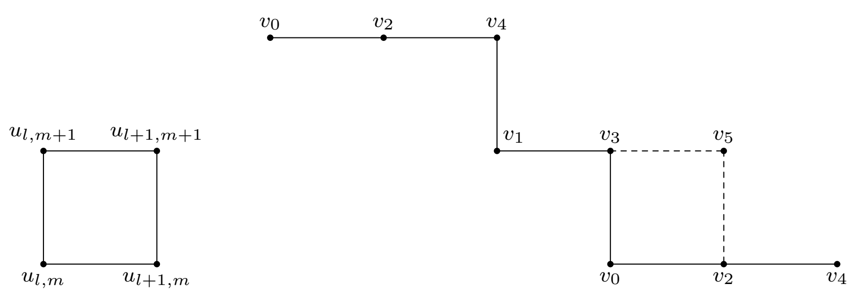

Fig. 1. Quad equation and the $(3, -2)$ reduction.

2.3. Lagrangian Equation and Ostrogradsky Transformation

Given a mapping obtained from an Euler–Lagrange equation, Bruschi et al. presented a transformation to rewrite this map in a canonical form [5]. We use this canonical form to derive a Poisson structure for mappings obtained from the Euler–Lagrange equation.

We start with the following symplectic map without knowing its symplectic structure

$$\phi : x(n) \mapsto x(n + 1),$$

where $x(n), x(n + 1) \in \mathbb{R}^{2N}$. We rewrite the map in canonical form as follows:

$$\big(q(n), p(n)\big) \mapsto \big(q(n + 1), p(n + 1)\big), \tag{2.6}$$

with $q_j(n + 1) = g_j\big(q(n), p(n)\big)$ and $p_j(n + 1) = f_j\big(q(n), p(n)\big)$ and such that $\{f_i, f_j\} = \{g_i, g_j\} = 0$ and $\{f_i, g_j\} = \delta_{ij}$ for $1 \leqslant i, j \leqslant N$.

Suppose we have a Lagrangian $L\big(u(k), u(k + 1), \ldots, u(k + N)\big)$ and the discrete action functional

$$I = \sum_{k \in \mathbb{Z}} L\big(u(k), u(k + 1), \ldots, u(k + N)\big).$$

The discrete Euler – Lagrange equation for the Lagrangian L is

$$\frac{\delta I}{\delta u(n)} = 0,$$

which means

$$\frac{\partial L\big(u(n), u(n+1), \ldots, u(n+N)\big)}{\partial u(n)} + \frac{\partial L\big(u(n-1), u(n), \ldots, u(n+N-1)\big)}{\partial u(n)} + \ldots + \frac{\partial L\big(u(n-N), u(n-N+1), \ldots, u(n)\big)}{\partial u(n)} = 0.$$

We denote $u^{(s)} = u(n+s)$ and $L_r = \frac{\partial L}{\partial u^{(r)}}$ and introduce a shift operator $\mathcal{E}$, i. e, $\mathcal{E}^j\,(u(n)) = u(n+j)$ for $j \in \mathbb{Z}$. The Euler – Lagrange equation can be rewritten as follows:

$$\sum_{r=0}^{N} \mathcal{E}^{-r} L_r = 0.$$

Definition 2. *A Lagrangian L is called normal if $L_{0N} = \frac{\partial^2 L}{\partial u^{(0)} u^{(N)}} \neq 0$.*

Given a normal Lagrangian L, we obtain a (nonlinear) difference equation of order $2N$. We introduce coordinates q_i and p_i, so that we can write the map derived from the Euler – Lagrange equation in canonical form. For example, we introduce a transformation

$$\big(u^{(-N)}, \ldots, u^{(0)}, \ldots, u^{(N-1)}\big) \mapsto (q_1, q_2, \ldots, q_N, p_1, p_2, \ldots, p_N),$$

where

$$\begin{aligned} q_i &= u^{(i-1)} = u(n+i-1), \\ p_i &= \mathcal{E}^{-1} \sum_{k=0}^{N-i} \mathcal{E}^{-k} L_{k+i}, \end{aligned} \tag{2.7}$$

for $i = 1, 2, \ldots, N$. This transformation can be considered as an analog of Ostrogradsky's transformation, cf. [8]. We have, using (2.7),

$$p_1 = \sum_{k=1}^{N} \mathcal{E}^{-k} L_k = -L_0\big(q_1, q_2, \ldots, q_N, u^{(N)}\big),$$

where we suppose that we can obtain

$$u^{(N)} = \alpha(q, p_1).$$

We give a proof for this result.

Lemma 1. *The map (2.6) has the following canonical form:*

$$q_i(n+1) = g_i(q,p) = q_{i+1}(n), \;\; i = 1, 2, \ldots, N-1, \tag{2.8}$$

$$q_N(n+1) = g_N(q,p) = \alpha(q, p_1), \tag{2.9}$$

$$\begin{aligned} p_i(n+1) = f_i(q,p) = p_{i+1}(n) + \widetilde{L}_i(q, p_1), \;\; i = 1, 2, \ldots, N-1, \\ p_N(n+1) = f_N(q,p) = \widetilde{L}_N(q, p_1), \end{aligned} \tag{2.10}$$

where we denote $\widetilde{L}(q, p_1) = L\big(q_1, q_2, \ldots, q_N, \alpha(q_1, q_2, \ldots, q_N, p_1)\big)$.

Proof. It is easy to obtain (2.8), (2.9) and (2.10). For $1 \leqslant i \leqslant N-1$, we have

$$\begin{aligned} p_i(n+1) &= \sum_{k=0}^{N-i} \mathcal{E}^{-k} L_{k+i} = \sum_{k=1}^{N-i} \mathcal{E}^{-k} L_{k+i} + \widetilde{L}_i = \sum_{k=0}^{N-i-1} \mathcal{E}^{-k-1} L_{k+i+1} + \widetilde{L}_i \\ &= p_{i+1}(n) + \widetilde{L}_i. \end{aligned}$$

Now we prove that the mapping (2.6) preserves the canonical symplectic structure, i.e., we show that $\{f_i, f_j\} = \{g_i, g_j\} = 0$ and $\{f_i, g_j\} = \delta_{ij}$. For $1 \leqslant i, j \leqslant N-1$, we have

$$\{g_i, g_j\} = \{q_{i+1}, q_{j+1}\} = 0,$$

$$\{g_i, g_N\} = \{q_{i+1}, \widetilde{L}_N(q, p_1)\} = -\frac{\partial \widetilde{L}_N(q, p_1)}{\partial p_{i+1}} = 0,$$

$$\{f_i, g_j\} = \{p_{i+1}, q_{j+1}\} + \{\widetilde{L}_i(q, p_1), q_{j+1}\} = \delta_{i+1j+1} + \frac{\partial \widetilde{L}_i(q, p_1)}{\partial p_{j+1}} = \delta_{ij},$$

$$\begin{aligned}\{f_N, g_N\} &= \{\widetilde{L}_N(q, p_1), \alpha_1(q, p_1)\} = \frac{\partial \widetilde{L}_N(q, p_1)}{\partial p_1}\frac{\partial \alpha(q, p_1)}{\partial q_1} - \frac{\partial \widetilde{L}_N(q, p_1)}{\partial q_1}\frac{\partial \alpha(q, p_1)}{\partial p_1} \\ &= \left(L_{NN}\frac{\partial u^{(N)}}{\partial p_1}\frac{\partial u^{(N)}}{\partial q_1} - \big(L_{NN}\frac{\partial u^{(N)}}{\partial q_1} + L_{0N}\big)\frac{\partial u^{(N)}}{\partial p_1}\right)\Bigg|_{u^{(0)}=q_1,\ldots,u^{(N-1)}=q_N,u^{(N)}=\alpha(q,p_1)} \\ &= -L_{N0}\frac{\partial u^{(N)}}{\partial p_1} = \frac{\partial p_1}{\partial p_1} = 1\end{aligned}$$

$$\begin{aligned}\{f_i, f_j\} &= \{p_{i+1}, p_{j+1}\} + \{p_{i+1}, \widetilde{L}_j(q, p_1)\} + \{\widetilde{L}_i(q, p_1), p_{j+1}\} + \{\widetilde{L}_i(q, p_1), \widetilde{L}_j(q, p_1)\} \\ &= \frac{\partial \widetilde{L}_j(q, p_1)}{\partial q_{i+1}} - \frac{\partial \widetilde{L}_i(q, p_1)}{\partial q_{j+1}} + \frac{\partial \widetilde{L}_i}{\partial p_1}\frac{\partial \widetilde{L}_j}{\partial q_1} - \frac{\partial \widetilde{L}_i}{\partial q_1}\frac{\partial \widetilde{L}_j}{\partial p_1} \\ &= \Bigg(L_{ij} + L_{Nj}\frac{\partial u^{(N)}}{\partial q_{i+1}} - L_{ji} - L_{Ni}\frac{\partial u^{(N)}}{\partial q_{j+1}} + L_{Ni}\frac{\partial u^{(N)}}{\partial p_1}\big(L_{0j} + L_{Nj}\frac{\partial u^{(N)}}{\partial q_1}\big) \\ &\quad - \big(L_{0i} + L_{Ni}\frac{\partial u^{(N)}}{\partial q_1}\big)L_{Nj}\frac{\partial u^{(N)}}{\partial p_1}\Bigg)\Bigg|_{u^{(0)}=q_1,\ldots,u^{(N-1)}=q_N,u^{(N)}=\alpha(q,p_1)} \\ &= \Bigg(L_{Nj}\frac{\partial u^{(N)}}{\partial q_{i+1}} - L_{Ni}\frac{\partial u^{(N)}}{\partial q_{j+1}} + L_{Ni}\frac{\partial u^{(N)}}{\partial p_1}L_{0j} \\ &\quad - L_{0i}L_{Nj}\frac{\partial u^{(N)}}{\partial p_1}\Bigg)\Bigg|_{u^{(0)}=q_1,\ldots,u^{(N-1)}=q_N,\ldots,u^{(N)}=\alpha(q,p_1)}\end{aligned}$$

$$\begin{aligned}\{f_i, f_N\} &= \{p_{i+1}, \widetilde{L}_N(q, p_1)\} + \{\widetilde{L}_i(q, p_1), \widetilde{L}_N(q, p_1)\} \\ &= \frac{\widetilde{L}_N(q, p_1)}{q_{i+1}} + \frac{\partial \widetilde{L}_i}{\partial p_1}\frac{\partial \widetilde{L}_N}{\partial q_1} - \frac{\partial \widetilde{L}_i}{\partial q_1}\frac{\partial \widetilde{L}_N}{\partial p_1} \\ &= \left(L_{iN} + L_{NN}\frac{\partial u^{(N)}}{\partial q_{i+1}} + L_{iN}L_{0N}\frac{\partial u^{(N)}}{\partial p_1} - L_{i0}L_{NN}\frac{\partial u^{(N)}}{\partial p_1}\right)\Bigg|_{u^{(0)}=q_1,u^{(N-1)}=q_N,u^{(N)}=\alpha(q,p_1)}.\end{aligned}$$

We have $p_1 = -L_0\,|_{u^{(0)}=q_1,\ldots,u^{(N-1)}=q_N,u^{(N)}=\alpha(q,p_1)}$, therefore

$$\begin{aligned}1 &= -L_{0N}\frac{\partial u^{(N)}}{\partial p_1}\Big|_{u^{(0)}=q_1,\ldots,u^{(N-1)}=q_N,u^{(N)}=\alpha(q,p_1)}, \\ 0 &= \left(-L_{0i} - L_{0N}\frac{\partial u^{(N)}}{\partial q_{i+1}}\right)\Bigg|_{u^{(0)}=q_1,\ldots,u^{(N-1)}=q_N,u^{(N)}=\alpha(q,p_1)}.\end{aligned}$$

Thus, we get $\frac{\partial u^{(N)}}{\partial q_{i+1}} = L_{0i}\frac{\partial u^{(N)}}{\partial p_1}$. This implies that $\{f_i, f_j\} = 0$ and $\{f_i, f_N\} = 0$. □

3. POISSON BRACKETS FOR $(q, -p)$ REDUCTIONS OF THE KdV AND THE PLUS-KdV EQUATIONS

The KdV equation and a similar equation which we call the plus-KdV equation are given as follows:

$$u_{l+1,m} - u_{l,m+1} = \frac{1}{u_{l,m}} - \frac{1}{u_{l+1,m+1}}, \tag{3.1}$$

$$u_{l+1,m} + u_{l,m+1} = \frac{1}{u_{l,m}} + \frac{1}{u_{l+1,m+1}}, \tag{3.2}$$

respectively. It is known that KdV can be obtained from a double copy of potential KdV equation or H_1 by introducing the new variable v, where $u_{l,m} = v_{l-1,m-1} - v_{l,m}$. Similarly, for plus-KdV, we introduce variable v such that $u_{l,m} = v_{l-1,m-1} + v_{l,m}$. In terms of variables v, Eqs. (3.1) and (3.2) become

$$(v_{l+1,m} + v_{l-1,m}) - (v_{l,m-1} + v_{l,m+1}) + \frac{1}{v_{l-1,m-1} - v_{l,m}} - \frac{1}{v_{l,m} - v_{l+1,m+1}} = 0,$$

$$v_{l+1,m} + v_{l-1,m} + v_{l,m-1} + v_{l,m+1} - \frac{1}{v_{l-1,m-1} + v_{l,m}} - \frac{1}{v_{l,m} + v_{l+1,m+1}} = 0$$

These equations are Euler – Lagrange equations with the corresponding Lagrangians

$$L^1_{l,m} = v_{l,m}v_{l+1,m} - v_{l,m}v_{l,m+1} - \ln|v_{l,n} - v_{l+1,m+1}|,$$

$$L^2_{l,m} = v_{l,m}v_{l+1,m} + v_{l,m}v_{l,m+1} - \ln|v_{l,n} + v_{l+1,m+1}|$$

The $(q, -p)$ reduction where $\gcd(p, q) = 1$ and $p < q$ gives us the following Lagrangians:

$$L^1_n = v_n v_{n+p} - v_n v_{n+q} - \ln|v_n - v_{n+p+q}|, \tag{3.3}$$

$$L^2_n = v_n v_{n+p} + v_n v_{n+q} - \ln|v_n + v_{n+p+q}|.$$

The Euler – Lagrange equations derived from these Lagrangians are

$$v_{n+p} + v_{n-p} - (v_{n+q} + v_{n-q}) - \frac{1}{v_n - v_{n+p+q}} + \frac{1}{v_{n-p-q} - v_n} = 0,$$

$$v_{n+p} + v_{n-p} + (v_{n+q} + v_{n-q}) - \frac{1}{v_n + v_{n+p+q}} - \frac{1}{v_{n-p-q} + v_n} = 0,$$

which are the $(q, -p)$ reduction of KdV and plus-KdV via reduction $u_n = v_{n-p-q} - v_n$. In canonical coordinates, for a double copy of KdV we have

$$Q_i = v_{n+i-1}, \ \text{for } 1 \leqslant i \leqslant p+q,$$

$$P_1 = -\left(v_{n+p} - v_{n+q} - \frac{1}{v_n - v_{n+p+q}}\right) = v_{n-p} - v_{n-q} + \frac{1}{v_{n-p-q} - v_n},$$

$$P_i = v_{n+i-p-1} - v_{n+i-q-1} + \frac{1}{v_{n+i-p-q-1} - v_{n+i-1}}, \ \text{for } 1 < i \leqslant p,$$

$$P_i = -v_{n+i-q-1} + \frac{1}{v_{n+i-p-q-1} - v_{n+i-1}}, \ \text{for } p < i \leqslant q,$$

$$P_i = \frac{1}{v_{n+i-p-q-1} - v_{n+i-1}}, \ \text{for } q < i \leqslant p+q.$$

For the plus-KdV equation, we have

$$Q_i = v_{n+i-1}, \ \text{for } 1 \leqslant i \leqslant p+q,$$

$$P_1 = -\left(v_{n+p} + v_{n+q} - \frac{1}{v_n + v_{n+p+q}}\right) = v_{n-p} + v_{n-q} - \frac{1}{v_{n-p-q} + v_n},$$

$$P_i = v_{n+i-p-1} + v_{n+i-q-1} - \frac{1}{v_{n+i-p-q-1} + v_{n+i-1}}, \text{ for } 1 < i \leqslant p,$$
$$P_i = v_{n+i-q-1} - \frac{1}{v_{n+i-p-q-1} + v_{n+i-1}}, \text{ for } p < i \leqslant q,$$
$$P_i = -\frac{1}{v_{n+i-p-q-1} + v_{n+i-1}}, \text{ for } q < i \leqslant p+q.$$

Using the canonical Poisson bracket for these variables, we obtain Poisson brackets for mappings obtained as the $(q, -p)$ reductions of Eqs. (3.1) and (3.2). For both equations, we distinguish between two cases where $p = 1$ and $p > 1$.

Theorem 1. *The KdV map admits the following Poisson structure for* $0 \leqslant i < j \leqslant p+q-1$.

- *If* $p = 1$*, we have*

$$\{u_{n+i}, u_{n+j}\} = \begin{cases} (-1)^{j-i} \prod_{r=i}^{j} u_{n+i}^2, & 0 < j-i < p+q-1, \\ \left(1 + (-1)^{p+q-1} \prod_{r=i+1}^{j-1} u_{n+r}^2\right) u_{n+i}^2 u_{n+j}^2, & j - i = p+q-1. \end{cases}$$

- *If* $p > 1$*, we have*

$$\{u_{n+i}, u_{n+j}\} = \begin{cases} u_{n+i}^2 u_{n+j}^2 & \text{if } j - i = q, \\ (-1)^{(j-i)/p} \prod_{k=0}^{(j-i)/p} u_{n+i+kp}^2, & \text{if } j - i \equiv 0 \mod p, \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

Proof. We note that, for the case $p = 1$, the Poisson bracket was given in [11]. We just need to prove this theorem for the case where $p > 1$.

We first need to prove that the Poison bracket (3.4) is preserved under the KdV map

$$\phi: \ (u_n, u_{n+1}, \ldots, u_{n+p+q-1}) \mapsto (u_{n+1}, u_{n+2}, \ldots, u_{n+p+q-1}, u_{n+p+q}),$$

where

$$u_{n+p+q} = \frac{-u_n}{u_n u_{n+p} - u_n u_{n+q} - 1}.$$

By direct calculation, we have

$$\frac{\partial u_{n+p+q}}{\partial u_n} = -\frac{u_{n+p+q}^2}{u_n^2}, \ \frac{\partial u_{n+p+q}}{\partial u_{n+p}} = u_{n+p+q}^2, \ \frac{\partial u_{n+p+q}}{\partial u_{n+q}} = -u_{n+p+q}^2.$$

It is easy to see that, for $0 \leqslant i < j < p+q-1$, we have

$$\{u_{n+i}^{'}, u_{n+j}^{'}\} = \{u_{n+i+1}, u_{n+j+1}\} = \{u_{n+i}, u_{n+j}\}|_{\mathbf{u}=\phi(\mathbf{u})}.$$

For $0 \leqslant i < j = p+q-1$, we have

$$\begin{aligned} \{u_{n+i}^{'}, u_{n+p+q-1}^{'}\} &= \{u_{n+i+1}, u_{n+p+q}\} \\ &= \{u_{n+i+1}, u_n\} \frac{u_{n+p+q}^2}{u_n^2} + \{u_{n+i+1}, u_{n+p}\} u_{n+p+q}^2 - \{u_{n+i+1}, u_{n+q}\} u_{n+p+q}^2. \end{aligned} \tag{3.5}$$

By considering whether $p|(i+1)$, one can see easily that if $i+1 \neq p$ and $i+1 \neq q$, then

$$\{u_{n+i+1}, u_n\} \frac{u_{n+p+q}^2}{u_n^2} + \{u_{n+i+1}, u_{n+p}\} u_{n+p+q}^2 = 0. \tag{3.6}$$

We consider 3 cases.

- Case 1 where $p+q-i-1 \neq q$ and $p \nmid (p+q-i-1)$. This means $i+1 \neq p$ and $p \nmid q-i-1$. Therefore, $\{u_{n+i+1}, u_{n+q}\} = 0$ and $i+1 \neq q$, $|i+1-p| \neq 0,$ and q. This also gives us (3.6). Thus, $\{u'_{n+i}, u'_{n+p+q-1}\} = 0 = \{u_{n+i}, u_{n+p+q-1}\}|_{\mathbf{u}=\phi(\mathbf{u})}$.

- Case 2 where $p+q-i-1 = q$, i. e., $i+1 = p$. It implies that the second term and third term in (3.5) vanish. Hence, we have

$$\begin{aligned}\{u'_{n+p-1}, u'_{n+p+q-1}\} &= \{u_{n+p}, u_n\}\frac{u^2_{n+p+q}}{u^2_n} = u^2_{n+p}u^2_n\frac{u^2_{n+p+q}}{u^2_n} = u^2_{n+p}u^2_{n+p+q}\\ &= u^2_{n+p-1}u^2_{n+p+q-1}|_{\mathbf{u}=\phi(\mathbf{u})} = \{u_{n+p-1}, u_{n+p+q-1}\}|_{\mathbf{u}=\phi(\mathbf{u})}.\end{aligned}$$

- Case 3 where $p|(p+q-i-1)$, i. e., $p|(q-i-1)$. If $i+1=q$, the second and third term in (3.5) equal to 0. Therefore, we have

$$\begin{aligned}\{u'_{n+q-1}, u'_{n+p+q-1}\} &= \{u_{n+q}, u_n\}\frac{u^2_{n+p+q}}{u^2_n} = -u^2_{n+q}u^2_{n+p+q}\\ &= -u^2_{n+q-1}u^2_{n+p+q-1}|_{\mathbf{u}=\phi(\mathbf{u})} = \{u_{n+q-1}, u_{n+p+q-1}\}|_{\mathbf{u}=\phi(\mathbf{u})}.\end{aligned}$$

If $i+1 \neq q$ and $i+1 \neq p$, we have (3.6). Since $i+1-q < p$, we have $i+1-q < 0$. Thus, (3.5) becomes

$$\begin{aligned}\{u'_{n+i}, u'_{n+p+q-1}\} &= \{u_{n+i+1}, u_{n+q}\}u^2_{n+p+q} = -(-1)^{(q-i-1)/p}u^2_{n+i+1}u^2_{n+i+1+p}u_{n+q}u_{n+q+p^2}\\ &= (-1)^{(p+q-i-1)/p}\prod_{k=0}^{(p+q-i-1)/p} u^2_{n+i+1+kp} = \{u_{n+i-1}, u_{n+p+q-1}\}|_{\mathbf{u}=\phi(\mathbf{u})}.\end{aligned}$$

To complete this proof, we need to prove the Jacobian identity (2.4):

$$\sum_l \left(\Omega_{li}\frac{\partial}{\partial u_{n+l}}\Omega_{jk} + \Omega_{lj}\frac{\partial}{\partial u_{n+l}}\Omega_{ki} + \Omega_{lk}\frac{\partial}{\partial u_{n+l}}\Omega_{ij}\right) = 0, \ \forall i,j,k. \tag{3.7}$$

Without loss of generality, we can assume that $0 \leqslant i \leqslant j \leqslant k \leqslant p+q-1$. It is easy to see that the Jacobian identity holds if two indices in (i,j,k) are equal. We need to prove the case where $0 \leqslant i < j < k \leqslant p+q-1$. We also distinguish 3 cases.

- Case 1 where $j-i=q$. Since $k < p+q$, we have $0 < k-j < p$. Therefore, $\Omega_{jk} = 0$. Furthermore, one can see easily that

$$\sum_{l=0}^{p+q-1}\Omega_{lk}\frac{\partial\Omega_{ij}}{\partial u_{n+l}} = \frac{2\Omega_{ik}\Omega_{ij}}{u_{n+i}} + \frac{2\Omega_{jk}\Omega_{ij}}{u_{n+j}} = \frac{2\Omega_{ik}\Omega_{ij}}{u_{n+i}}.$$

If $p \nmid (k-i)$, then $\Omega_{ki} = 0$. Thus, the Jacobian identity (3.7) holds. If $p|(k-i)$, then we have

$$\sum_{l=0}^{p+q-1}\Omega_{lj}\frac{\partial\Omega_{ki}}{\partial u_{n+l}} = \sum_{i<l\leqslant k,\ l\equiv i \mod p}\frac{2\Omega_{lj}\Omega_{ki}}{u_{n+l}} + \frac{2\Omega_{ij}\Omega_{ki}}{u_{n+i}} = -\frac{2\Omega_{ij}\Omega_{ik}}{u_{n+i}},$$

where we have used $\Omega_{lj} = 0$ as $p \nmid (j-l)$ and $j-l \neq \pm q$ for $i < l \leqslant k$ and $p|(l-i)$.

- Case 2 where $p|(j-i)$. We have

$$\sum_{l=0}^{p+q-1}\Omega_{lk}\frac{\partial\Omega_{ij}}{\partial u_{n+l}} = \sum_{i\leqslant l\leqslant j,\ l\equiv i \mod p}\frac{2\Omega_{lk}\Omega_{ij}}{u_{n+l}}. \tag{3.8}$$

Since $p \leqslant j-i$, we have $k-j<q$. Thus, if $k-i=q$, then $p \nmid (k-j)$. This yields $\Omega_{jk}=0$ and

$$\sum_{l=0}^{p+q-1} \Omega_{lj} \frac{\partial \Omega_{ki}}{\partial u_{n+l}} = \frac{2\Omega_{ij}\Omega_{ki}}{u_{n+i}} + \frac{2\Omega_{kj}\Omega_{ki}}{u_{n+k}} = \frac{2\Omega_{ij}\Omega_{ki}}{u_{n+i}}.$$

However, for $i<l \leqslant j$ and $l \equiv i \mod p$, we have $\Omega_{lk}=0$. Therefore, the RHS of (3.8) becomes $2\Omega_{ik}\Omega_{ij}/u_{n+i}$. Hence, we obtain the Jacobian identity (3.7).

If $k-i \neq q$ and $p \nmid k-i$, then $\Omega_{jk}=\Omega_{ki}=\Omega_{lk}=0$ for $i<l<j$ and $l \equiv i \mod p$ (as $k-l< p+q-(i+p) \leqslant q$ and $p \nmid (k-l)$. This implies that LHS (3.8) $=0$. Thus, the Jacobian identity (3.7) holds.

If $p|(k-i)$, i. e., $i \equiv j \equiv k \mod p$, then the LHS of the Jacobian identity (3.7) is

$$LHS\ (3.7) = \sum_{j \leqslant l \leqslant k,\ l \equiv i \mod p} \frac{2\Omega_{li}\Omega_{jk}}{u_{n+l}} + \sum_{i \leqslant l \leqslant k,\ l \equiv i \mod p} \frac{2\Omega_{lj}\Omega_{ki}}{u_{n+l}} + \sum_{i \leqslant l \leqslant j,\ l \equiv i \mod p} \frac{2\Omega_{lk}\Omega_{ij}}{u_{n+l}}.$$

It is easy to see that the three terms in this sum can be written as follows:

$$\sum_{j \leqslant l \leqslant k,\ l \equiv i \mod p} \frac{2\Omega_{li}\Omega_{jk}}{u_{n+l}} = - \sum_{j < l \leqslant k,\ l \equiv i \mod p} \frac{2\Omega_{ik}\Omega_{jl}}{u_{n+l}} + \frac{2\Omega_{ji}\Omega_{jk}}{u_{n+j}},$$

$$\sum_{i \leqslant l \leqslant k,\ l \equiv i \mod p} \frac{2\Omega_{lj}\Omega_{ki}}{u_{n+l}} = \sum_{i \leqslant l < j,\ l \equiv i \mod p} \frac{2\Omega_{lj}\Omega_{ki}}{u_{n+l}} + \sum_{j < l \leqslant k,\ l \equiv i \mod p} \frac{2\Omega_{lj}\Omega_{ki}}{u_{n+l}},$$

$$\sum_{i \leqslant l \leqslant j,\ l \equiv i \mod p} \frac{2\Omega_{lk}\Omega_{ij}}{u_{n+l}} = \sum_{i \leqslant l < j,\ l \equiv i \mod p} \frac{2\Omega_{ik}\Omega_{lj}}{u_{n+l}} + \frac{2\Omega_{jk}\Omega_{ij}}{u_{n+j}}.$$

Adding these three terms together, we get LHS (3.7) $=0$.

- Case 3 where $j-i \neq q$ and $p \nmid (j-i)$, i. e., we have $\Omega_{ij}=0$. Similarly, by considering the following subcases $k-j=q$; $p|(k-j)$; $k-j \neq q$, and$p \nmid (k-j)$, we can prove that the Jacobian identity (3.7) holds.

□

Analogously, we can derive the following result for the map obtained from the plus-KdV.

Theorem 2. *The map obtained as the* $(q,-p)$ *reduction of Eq.* (3.2) *admits the following Poisson structure for* $0 \leqslant i < j \leqslant p+q-1$.

- *If* $p=1$, *we have*

$$\{u_{n+i}, u_{n+j}\} = \begin{cases} (-1)^{j-i} \prod_{r=i}^{j} u_{n+i}^2, & 0<j-i<p+q-1, \\ \big(-1+(-1)^{p+q-1}\prod_{r=i+1}^{j-1} u_{n+r}^2\big) u_{n+i}^2 u_{n+j}^2, & j-i=p+q-1. \end{cases}$$

- *If* $p>1$, *we have*

$$\{u_{n+i}, u_{n+j}\} = \begin{cases} -u_{n+i}^2 u_{n+j}^2 & \textit{if } j-i=q, \\ (-1)^{(j-i)/p} \prod_{k=0}^{(j-i)/p} u_{n+i+kp}^2, & \textit{if } j-i \equiv 0 \mod p, \\ 0 & \textit{otherwise}. \end{cases} \tag{3.9}$$

3.1. Examples

The matrices

$$L = \begin{pmatrix} u_{l,m} - \frac{1}{u_{l+1,m}} & \lambda \\ 1 & 0 \end{pmatrix}, \qquad M = \begin{pmatrix} u_{l,m} & \lambda \\ 1 & \frac{1}{u_{l,m}}, \end{pmatrix}$$

are Lax-matrices for the KdV equation (3.1). For the $(3,-2)$-reduction (with $n = 2l + 3m$, $u_{l,m} \to v_n$, $L_{l,m} \to L_n$, $M_{l,m} \to M_n$),

$$v_{n+2} - v_{n+3} = \frac{1}{v_n} - \frac{1}{v_{n+5}},$$

the trace of the monodromy matrix $\mathcal{L} = M_0^{-1} L_1 M_1^{-1} L_2 L_0$ yields the following three invariants:

$$I_1 = \frac{1}{v_0} + \frac{1}{v_1} + \frac{1}{v_2} + \frac{1}{v_3} + \frac{1}{v_4} - v_2, \quad I_2 = \frac{(v_2v_4-1)(v_1v_3-1)(v_0v_2-1)}{v_0v_1v_2v_3v_4}, \quad I_3 = \frac{(v_1v_4+1)(v_0v_3+1)}{v_0v_1v_2v_3v_4}.$$

We have

$$\Omega = \begin{pmatrix} 0 & 0 & -v_0^2v_2^2 & v_0^2v_3^2 & v_0^2v_2^2v_4^2 \\ 0 & 0 & 0 & -v_1^2v_3^2 & v_1^2v_4^2 \\ v_0^2v_2^2 & 0 & 0 & 0 & -v_2^2v_4^2 \\ -v_0^2v_3^2 & v_1^2v_3^2 & 0 & 0 & 0 \\ -v_0^2v_2^2v_4^2 & -v_1^2v_4^2 & v_2^2v_4^2 & 0 & 0 \end{pmatrix},$$

and I_1 is a Casimir of the corresponding bracket. The rank of the bracket is 4, and we have $5 - 4/2 = 3$ functionally independent invariants in involution with respect to the Poisson bracket, so the Poisson map is completely integrable.

The matrices

$$L = \begin{pmatrix} u_{l,m} - \frac{1}{u_{l+1,m}} & \lambda \\ 1 & 0 \end{pmatrix}, \qquad M = \begin{pmatrix} u_{l,m} & \lambda \\ -1 & -\frac{1}{u_{l,m}}, \end{pmatrix}$$

are anti-Lax-matrices for the plus-KdV equation (3.2), which satisfy $L_{l,m+1}M_{l,m} \equiv -M_{l+1,m}L_{l,m}$ modulo the equation. The idea of an anti-Lax pair was suggested to one of the authors by J. A. G. Roberts [17]. For the $(3,-2)$-reduction,

$$v_{n+2} + v_{n+3} = \frac{1}{v_n} + \frac{1}{v_{n+5}},$$

the trace of the monodromy matrix gives rise to the following three 2-integrals:

$$J_1 = \frac{1}{v_0} - \frac{1}{v_1} + \frac{1}{v_2} - \frac{1}{v_3} + \frac{1}{v_4} - v_2, \quad J_2 = \frac{(v_2v_4-1)(v_1v_3-1)(v_0v_2-1)}{v_0v_1v_2v_3v_4}, \quad J_3 = \frac{(v_1v_4-1)(v_0v_3-1)}{v_0v_1v_2v_3v_4},$$

that is, we have $J_i(v_1, \ldots, v_5) \equiv -J_i(v_0, \ldots, v_4)$. We have

$$\Omega = \begin{pmatrix} 0 & 0 & -v_0^2v_2^2 & -v_0^2v_3^2 & v_0^2v_2^2v_4^2 \\ 0 & 0 & 0 & -v_1^2v_3^2 & -v_1^2v_4^2 \\ v_0^2v_2^2 & 0 & 0 & 0 & -v_2^2v_4^2 \\ v_0^2v_3^2 & v_1^2v_3^2 & 0 & 0 & 0 \\ -v_0^2v_2^2v_4^2 & v_1^2v_4^2 & v_2^2v_4^2 & 0 & 0 \end{pmatrix},$$

and J_1 is a Casimir of the corresponding bracket. The rank of the bracket is 4, and we have $5 - 4/2 = 3$ functionally independent invariants in involution, namely, $I_1 = J_1/J_2$, $I_2 = J_2/J_3$, and $I_3 = J_2J_3$, so the Poisson map is completely integrable.

4. POISSON BRACKETS FOR $(q,-p)$ REDUCTIONS OF THE LOTKA – VOLTERRA EQUATION, AND THE LIOUVILLE EQUATION

The discrete Lotka – Volterra equation is given as follows (cf. [12]):

$$u_{l,m+1}(1+u_{l,m}) = u_{l+1,m}(1+u_{l+1,m+1}). \tag{4.1}$$

We introduce the new variable $w_{l,m} = \ln u_{l,m}$ to get

$$w_{l,m+1} - w_{l+1,m} = -\ln(1+w_{l,m}) + \ln(1+w_{l+1,m+1}). \tag{4.2}$$

This equation's functional form is similar to that of the KdV equation. Therefore, it suggests that one could try to introduce $w_{l,m} = v_{l-1,m-1} - v_{l,m}$ to write Eq. (4.2) as an Euler – Lagrange equation. One can easily derive a Lagrangian

$$L = v_{l,m}v_{l,m+1} - v_{l,m}v_{l+1,m} + F(v_{l,m} - v_{l+1,m+1}), \tag{4.3}$$

where

$$F(x) = \int_0^x \ln(1+e^t)dt.$$

Using the discrete analogue of the Ostrogradsky transformation, we can derive Poisson brackets of mappings obtained by the $(p,-q)$ reduction of Eq. (4.2) in the v variable and then in the w variable. Once we have a Poisson bracket in the w variable, the associate Poisson bracket in the original variable is calculated by

$$\{u_{n+i}, u_{n+j}\} = \{e^{w_{n+i}}, e^{w_{n+j}}\} = e^{w_{n+i}}e^{w_{n+j}}\{w_{n+i}, w_{n+j}\}.$$

We obtain the following theorem.

Theorem 3. *The map obtained as the $(q,-p)$ reduction of Eq. (4.1) admits the following Poisson structure for $0 \leqslant i < j \leqslant p+q-1$.*

- *If $p = 1$, we have*

$$\{u_{n+i}, u_{n+j}\} = \begin{cases} (1+u_{n+i})(1+u_{n+j})\prod_{r=i+1}^{j-1}\left(\frac{1+u_{n+r}}{u_{n+r}}\right), & 0 < j-i < p+q-1, \\ (1+u_{n+i})(1+u_{n+j})\Big(\prod_{r=i+1}^{j-1}\left(\frac{1+u_{n+r}}{u_{n+r}}\right) - 1\Big), & j-i = p+q-1. \end{cases} \tag{4.4}$$

- *If $p > 1$, we have*

$$\{u_{n+i}, u_{n+j}\} = \begin{cases} -(u_{n+i}+1)(u_{n+j}+1) & \text{if } j-i = q, \\ (u_{n+i}+1)(u_{n+j}+1)\prod_{k=1}^{(j-i)/p-1}\left(\frac{u_{n+i+kp}+1}{u_{n+i+kp}}\right), & \text{if } j-i \equiv 0 \mod p, \\ 0 & \text{otherwise.} \end{cases} \tag{4.5}$$

Similarly, if we take

$$L = v_{l,m}v_{l,m+1} + v_{l,m}v_{l+1,m} + F(v_{l,m} + v_{l+1,m+1})$$

and $w_{l,m} = v_{l-1,m-1} + v_{l,m}$, we obtain a Poisson bracket for the $(q,-p)$ reduction of the following equation:

$$(1+u_{l,m})(1+u_{l+1,m+1})u_{l+1,m}u_{l,m+1} - 1 = 0. \tag{4.6}$$

Theorem 4. *The map obtained as the $(q,-p)$ reduction of Eq. (4.6) admits the following Poisson structure for $0 \leqslant i < j \leqslant p+q-1$.*

- *If $p = 1$, we have*

$$\{u_{n+i}, u_{n+j}\} = \begin{cases} (-1)^{j-i}(1+u_{n+i})(1+u_{n+j}) \prod_{r=i+1}^{j-1} \left(\frac{1+u_{n+r}}{u_{n+r}}\right), & 0 < j-i < p+q-1, \\ (1+u_{n+i})(1+u_{n+j})\Big((-1)^{j-i} \prod_{r=i+1}^{j-1} \left(\frac{1+u_{n+r}}{u_{n+r}}\right) - 1\Big), & j-i = p+q-1. \end{cases} \tag{4.7}$$

- *If $p > 1$, we have*

$$\{u_{n+i}, u_{n+j}\} = \begin{cases} -(u_{n+i}+1)(u_{n+j}+1) & \textit{if } j-i = q, \\ (-1)^{(j-i)/p}(u_{n+i}+1)(u_{n+j}+1) \prod_{k=1}^{(j-i)/p-1} \Big(\frac{u_{n+i+kp}+1}{u_{n+i+kp}}\Big), & \textit{if } j-i \equiv 0 \mod p, \\ 0 & \textit{otherwise.} \end{cases} \tag{4.8}$$

If we take

$$L = v_{l,m}v_{l,m+1} + v_{l,m}v_{l+1,m} - F(v_{l,m} + v_{l+1,m+1}),$$

we obtain Poisson brackets of maps obtained as reductions of the discrete Liouville equation

$$(1+u_{l,m})(1+u_{l+1,m+1}) - u_{l+1,m}u_{l,m+1} = 0. \tag{4.9}$$

Theorem 5. *The map obtained as the $(q, -p)$ reduction of (4.9) admits the following Poisson structure for $0 \leqslant i < j \leqslant p+q-1$.*

- *If $p = 1$, we have*

$$\{u_{n+i}, u_{n+j}\} = \begin{cases} -(1+u_{n+i})(1+u_{n+j}) \prod_{r=i+1}^{j-1} \left(\frac{1+u_{n+r}}{u_{n+r}}\right), & 0 < j-i < p+q-1, \\ -(1+u_{n+i})(1+u_{n+j})\Big(\prod_{r=i+1}^{j-1} \left(\frac{1+u_{n+r}}{u_{n+r}}\right) + 1\Big), & j-i = p+q-1. \end{cases} \tag{4.10}$$

- *If $p > 1$, we have*

$$\{u_{n+i}, u_{n+j}\} = \begin{cases} -(u_{n+i}+1)(u_{n+j}+1) & \textit{if } j-i = q, \\ -(u_{n+i}+1)(u_{n+j}+1) \prod_{k=1}^{(j-i)/p-1} \Big(\frac{u_{n+i+kp}+1}{u_{n+i+kp}}\Big), & \textit{if } j-i \equiv 0 \mod p, \\ 0 & \textit{otherwise.} \end{cases} \tag{4.11}$$

Remark 1. We note that all equations that we have considered in this paper can be written in the following form (after some transformations):

$$u_{l+1,m} - u_{l,m+1} = G(u_{l,m}) - G(u_{l+1,m+1}) \text{ or } u_{l+1,m} + u_{l,m+1} = G(u_{l,m}) + G(u_{l+1,m+1}).$$

In fact, using the Ostrogradsky transformation and a Lagrangian which has the same form as (4.3), one can always find Poisson brackets for maps obtained as reductions of these equations. This method can also be applied to more general (q, p)-reductions, i. e., we do not require $\gcd(p, q) = 1$.

4.1. Examples

The matrices

$$L = \begin{pmatrix} 1 & -\frac{u_{l+1,m}}{\lambda^2} \\ \frac{1}{u_{l+1,m}+1} & \frac{u_{l+1,m}}{u_{l+1,m}+1} \end{pmatrix}, \qquad M = \begin{pmatrix} 1 & -\frac{u_{l,m+1}(u_{l+1,m}+1)}{\lambda^2} \\ 1 & 0, \end{pmatrix}$$

are Lax-matrices for the Lotka–Volterra equation (4.1). For the $(3,-2)$-reduction,

$$v_{n+3}(v_n+1) = v_{n+2}(v_{n+5}+1),$$

the determinant of the monodromy matrix provides the invariant

$$I_1 = \frac{(v_0+1)(v_1+1)(v_2+1)(v_3+1)(v_4+1)}{v_2},$$

which is also a Casimir of the Poisson bracket defined by

$$\Omega = \begin{pmatrix} 0 & 0 & (v_0+1)(v_2+1) & -(v_0+1)(v_3+1) & (v_0+1)(\frac{1}{v_2}+1)(v_4+1) \\ 0 & 0 & 0 & (v_1+1)(v_3+1) & -(v_1+1)(v_4+1) \\ -(v_0+1)(v_2+1) & 0 & 0 & 0 & (v_2+1)(v_4+1) \\ (v_0+1)(v_3+1) & -(v_1+1)(v_3+1) & 0 & 0 & 0 \\ -(v_0+1)(\frac{1}{v_2}+1)(v_4+1) & (v_1+1)(v_4+1) & -(v_2+1)(v_4+1) & 0 & 0 \end{pmatrix}.$$

The trace yields two additional functionally independent integrals

$$I_2 = v_0v_3 - v_2 + v_1v_4,$$

$$I_3 = (v_1+1)(v_0+v_2+2) + (v_3+1)(v_2+v_4+2) + \frac{(v_0+1)(v_1+v_3+1)(v_4+1)}{v_2}.$$

These integrals are in involution, which shows the mapping is completely integrable.

The $(3,-2)$-reduction of Eq. (4.6),

$$(1+v_n)v_{n+2}v_{n+3}(1+v_{n+5}) = 1,$$

is a Poisson map with Poisson structure

$$\Omega = \begin{pmatrix} 0 & 0 & -(v_0+1)(v_2+1) & -(v_0+1)(v_3+1) & (v_0+1)(\frac{1}{v_2}+1)(v_4+1) \\ 0 & 0 & 0 & -(v_1+1)(v_3+1) & -(v_1+1)(v_4+1) \\ (v_0+1)(v_2+1) & 0 & 0 & 0 & -(v_2+1)(v_4+1) \\ (v_0+1)(v_3+1) & (v_1+1)(v_3+1) & 0 & 0 & 0 \\ -(v_0+1)(\frac{1}{v_2}+1)(v_4+1) & (v_1+1)(v_4+1) & (v_2+1)(v_4+1) & 0 & 0 \end{pmatrix}.$$

The Casimir function

$$C = \frac{(v_0+1)v_2(v_2+1)(v_4+1)}{(v_1+1)(v_3+1)}$$

is a 2-integral, applying the map gives the multiplicative inverse. Hence the function $C + 1/C$ is an integral. We have not been able to find more integrals and, as the map seems to have nonvanishing entropy, we believe it is not integrable.

The $(3,-2)$-reduction of Eq. (4.9),

$$(1+v_n)(1+v_{n+5}) = v_{n+2}v_{n+3}, \tag{4.12}$$

is a Poisson map with Poisson structure

$$\Omega = \begin{pmatrix} 0 & 0 & -(v_0+1)(v_2+1) & -(v_0+1)(v_3+1) & -(v_0+1)(\frac{1}{v_2}+1)(v_4+1) \\ 0 & 0 & 0 & -(v_1+1)(v_3+1) & -(v_1+1)(v_4+1) \\ (v_0+1)(v_2+1) & 0 & 0 & 0 & -(v_2+1)(v_4+1) \\ (v_0+1)(v_3+1) & (v_1+1)(v_3+1) & 0 & 0 & 0 \\ (v_0+1)(\frac{1}{v_2}+1)(v_4+1) & (v_1+1)(v_4+1) & (v_2+1)(v_4+1) & 0 & 0 \end{pmatrix}.$$

The Casimir function

$$C = \frac{(v_0+1)(v_2+1)(v_4+1)}{(v_1+1)v_2(v_3+1)}$$

is a 2-integral with $C' = 1/C$ and so provides us with an integral $I_1 = C + 1/C$. Other k-integrals can be found from the i- and j-integrals given in [2] for an equation, [2, Eq. (19)], which is related to (4.9) by $u_{i,j} \to \frac{-1}{1+u_{l,m}}$. The reduction of these integrals yields the 2-integral

$$A = \frac{(v_0+v_3+1)(v_1+v_4+1)}{(v_1+1)(v_3+1)}$$

and the 3-integral

$$B = \frac{(v_0+v_2+1)(v_1+v_3+1)}{(v_0+1)(v_3+1)}.$$

Applying the map $v_i \mapsto v_{i+1}$ subject to (4.12), we obtain

$$A' = \frac{(v_0+v_3+1)v_2(v_1+v_4+1)}{(v_0+1)(v_2+1)(v_4+1)}, \quad A'' = A$$

and

$$B' = \frac{(v_1+v_3+1)(v_2+v_4+1)}{(v_1+1)(v_4+1)}, \quad B'' = \frac{(v_0+v_2+1)(v_2+v_4+1)}{v_2(v_2+1)}, \quad B''' = B.$$

We have $C = A/A'$ and it can be checked that the integrals I_1, $I_2 = A + A'$, $I_3 = BB'B''$ are functionally independent and in involution. The additional integral $I_4 = B + B' + B''$ is functionally independent and in involution with I_2 (and I_1), but not with I_3. A linearization of the lattice equation can be found in [2].

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REFERENCES

1. Adler, V. E., Bobenko, A. I., and Suris, Yu. B., Classification of Integrable Equations on Quad-Graphs: The Consistency Approach, *Comm. Math. Phys.,* 2003, vol. 233, no. 3, pp. 513–543.
2. Adler, V. E. and Startsev, S. Ya., On Discrete Analogues of the Liouville Equation, *Theoret. and Math. Phys.,* 1999, vol. 121, no. 2, pp. 1484–1495; see also: *Teoret. Mat. Fiz.,* 1999, vol. 121, no. 2, pp. 271–284.
3. Arnold, V. I., Kozlov, V. V., and Neishtadt, A. I., *Mathematical Aspects of Classical and Celestial Mechanics,* 3rd ed., Encyclopaedia Math. Sci., vol. 3, Berlin: Springer, 2006.
4. Błaszak, M., *Multi-Hamiltonian Theory of Dynamical Systems,* Texts Monogr. Phys., Berlin: Springer, 1998.
5. Bruschi, M., Ragnisco, O., Santini, P. M., and Tu, G. Zh., Integrable Symplectic Maps, *Phys. D,* 1991, vol. 49, no. 3, pp. 273–294.
6. Byrnes, G., Haggar, F. A., and Quispel, G. R. W., Sufficient Conditions for Dynamical Systems to Have Pre-Symplectic or Pre-Implectic Structures, *Phys. A,* 1999, vol. 272, nos. 1–2, pp. 99–129.

7. Capel, H. W., Nijhoff, F., and Papageorgiou, V. G., Complete Integrability of Lagrangian Mappings and Lattice KdV Type, *Phys. Lett. A,* 1991, vol. 155, nos. 5–6, pp. 377–387.
8. Dubrovin, B. A., Krichever, I. M., and Novikov, S. P., Integrable Systems: 1, in *Dynamical Systems 4,* V. I. Arnold, S. P. Novikov (Eds.), 2nd, exp. and rev. ed., Encyclopaedia Math. Sci., vol. 4, Berlin: Springer, 2001, pp. 177–332.
9. Emmrich, C. and Kutz, N., Doubly Discrete Lagrangian Systems Related to the Hirota and sine-Gordon Equation, *Phys. Lett. A,* 1995, vol. 201, nos. 2–3, pp. 156–160.
10. Faddeev, L. D. and Volkov, A. Y., Hirota Equation As an Example of Integrable Symplectic Map, *Lett. Math. Phys.,* 1994, vol. 32, no. 2, pp. 125–136.
11. Hone, A. N. W., van der Kamp, P. H., Quispel, G. R. W., and Tran, D. T., Integrability of Reductions of the Discrete Korteweg – de Vries and Potential Korteweg – de Vries Equations, *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.,* 2013, vol. 469, no. 2154, 20120747, 23 pp.
12. Levi, D. and Yamilov, R. I., The Generalized Symmetry Method for Discrete Equations, *J. Phys. A,* 2009, vol. 42, no. 45, 454012, 18 pp.
13. Maeda, Sh., Completely Integrable Symplectic Mapping, *Proc. Japan Acad. Ser. A Math. Sci.,* 1987, vol. 63, no. 6, pp. 198–200.
14. Olver, P. J., *Applications of Lie Groups to Differential Equations,* 2nd ed., Grad. Texts in Math., vol. 107, New York: Springer, 1993.
15. Papageorgiou, V. G., Nijhoff, F. W., and Capel, H. W., Integrable Mappings and Nonlinear Integrable Lattice Equations, *Phys. Lett. A,* 1990, vol. 147, nos. 2–3, pp. 106–114.
16. Quispel, G. R. W., Capel, H. W., Papageorgiou, V. G., and Nijhoff, F. W., Integrable Mappings Derived from Soliton Equations, *Phys. A,* 1991, vol. 173, nos. 1-2, pp. 243–266.
17. Roberts, J. A. G., *private communication* (2013).
18. Tran, D. T., Complete Integrability of Maps Obtained As Reductions of Integrable Lattice Equations, *PhD Thesis,* La Trobe Univ., Melbourne, 2011.
19. Tran, D. T., van der Kamp, P. H., and Quispel, G. R. W., Closed-Form Expressions for Integrals of Traveling Wave Reductions of Integrable Lattice Equations, *J. Phys. A,* 2009, vol. 42, no. 22, 225201, 20 pp.
20. Tran, D. T., van der Kamp, P. H., and Quispel, G. R. W., Sufficient Number of Integrals for the p^{th}-Order Lyness Equation, *J. Phys. A,* 2010, vol. 43, no. 30, 302001, 11 pp.
21. Tran, D. T., van der Kamp, P. H., and Quispel, G. R. W., Involutivity of Integrals of sine-Gordon, Modified KdV and Potential KdV Maps, *J. Phys. A,* 2011, vol. 44, no. 29, 295206, 13 pp.
22. van der Kamp, P. H., Initial Value Problems for Lattice Equations, *J. Phys. A,* 2009, vol. 42, no. 40, 404019, 16 pp.
23. van der Kamp, P. H. and Quispel, G. R. W., The Staircase Method: Integrals for Periodic Reductions of Integrable Lattice Equations, *J. Phys. A,* 2010, vol. 43, no. 46, 465207, 34 pp.
24. Veselov, A. P., Integrable Maps, *Russian Math. Surveys,* 1991, vol. 46, no. 5, pp. 1–51; see also: *Uspekhi Mat. Nauk,* 1991, vol. 46, no. 5, pp. 3–45.