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On proving integrability

Peter H van der Kamp

Faculty of Sciences, Division of Mathematics and Computer Science, Vrije Universiteit,
De Boelelaan 1081a, 1081 HV, Amsterdam, The Netherlands

E-mail: peter@few.vu.nl

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Abstract
We prove the conjecture, formulated in Foursov M V 2000 *Inverse Problems* **16** 259–74, that the system
$u_t = \frac{1}{2}u_3 + \frac{1}{2}v_3 + (2-\alpha)u_0u_1 + (6-\alpha)v_0u_1 + \alpha u_0v_1 + (4-\alpha)v_0v_1$
$v_t = \frac{1}{2}v_3 + \frac{1}{2}u_3 + (2-\alpha)v_0v_1 + (6-\alpha)u_0v_1 + \alpha v_0u_1 + (4-\alpha)u_0u_1$
has polynomial symmetries of order $2k$ and weight $2k+2n$ when $\alpha = 2(1-(k/n))$ for any non-negative integer k and any positive integer n. Moreover we prove the existence of infinitely many nonpolynomial symmetries for any α. This demonstrates the use of the implicit function theorem of Sanders and Wang together with the symbolic calculus of Gelfand and Dikiĭ to prove the existence of infinitely many symmetries of evolution equations.

1. Introduction

It was observed and conjectured (cf [5, 6, 9]) that the existence of one (or a few) symmetries implies the existence of infinitely many symmetries. Counterexamples were found in [1, 10] and a (p-adic) method to prove that the number of symmetries is finite has been developed (cf [2, 11]). These developments show that it is necessary to prove the existence of infinitely many symmetries. Although the methods employed in [3, 13–16] show how one can effectively obtain integrability proofs, still the observation and conjecture are used to argue that it is enough to find only one or two symmetries of a system in order to declare it integrable (cf [7, 12]). In this paper we explain and demonstrate the use of an implicit function theorem, as formulated in [14], and the symbolic calculus which is developed in [8].

In [7] a classification of third-order symmetrically coupled KdV-like equations with respect to the existence of two symmetries is presented. One system (4.7) in the list is quite special.

$$u_t = \tfrac{1}{2}u_3 + \tfrac{1}{2}v_3 + (2-\alpha)u_0u_1 + (6-\alpha)v_0u_1 + \alpha u_0v_1 + (4-\alpha)v_0v_1$$
$$v_t = \tfrac{1}{2}v_3 + \tfrac{1}{2}u_3 + (2-\alpha)v_0v_1 + (6-\alpha)u_0v_1 + \alpha v_0u_1 + (4-\alpha)u_0u_1.$$

For all values of α odd-order symmetries were found. At even order, symmetries were found as well, but only for some particular values of α. Foursov verified all weight two, four, six, eight and ten symmetries and formulated the following conjecture.

Conjecture 1 ([7]). *System 4.7 has symmetries of order $2k$ and weight $2k + 2n$ when $\alpha = 2(1 - (k/n))$ for any non-negative integer k and any positive integer n.*

A particular easy case is $\alpha = 2$; there are the symmetries of zero order and weight $2n$

$$u_t = (u - v)^n$$
$$v_t = -(u - v)^n.$$

No extra odd-weight symmetries were found because it was assumed that the symmetries were polynomial. The crucial observation one has to make is that the weight can be any number, i.e. the above system is a symmetry when $\alpha = 2$ for all $n \in \mathbb{C}$.

In this paper we prove that system 4.7 has infinitely many symmetries at any positive order for all $\alpha \neq 2$. The weight of the even-order symmetries is generally a real number (or complex when α is complex). Only for the special values of α stated in the conjecture does the weight become even at special orders. At $\alpha = 2$ there are symmetries at all odd orders and the symmetries of order zero but arbitrary weight. When $-2\alpha \in \mathbb{N}$ we find additional odd-order symmetries. A computer program that produces all these symmetries is included in appendix C. Also the existence of an extra set of symmetries of arbitrary order is proven and examples are given.

2. Implicit function theorem

We can view the right-hand side of an evolution equation $(u_t, v_t) = K$ as an element of a Lie algebra $\mathcal{L}$.

Definition 1. *An element $Q \in \mathcal{L}$ is called a generalized symmetry of K, or symmetry for short, if* $\mathrm{ad}(K)Q = [K, Q] = 0$. *An equation with infinitely many independent symmetries is said to be integrable and an infinite set of symmetries is called a hierarchy.*

The computation of symmetries can be very cumbersome. It is a useful procedure to divide the problem into a number of smaller computations. This can be done by introducing a filtration on the algebra.

Definition 2. *A Lie algebra $\mathcal{L}$ is filtered if $\mathcal{L} = \mathcal{L}^0 \supset \mathcal{L}^1 \supset \mathcal{L}^2 \supset \cdots$ such that $\cap_{i=0}^{\infty} \mathcal{L}^i = \{0\}$ and*

$$[\mathcal{L}^i, \mathcal{L}^j] \subset \mathcal{L}^{i+j}.$$

Now finding a symmetry of K is equivalent to solving the set of equations

$$[K, Q] \in \mathcal{L}^j \qquad \text{for } j = 1, 2, \ldots.$$

Under some conditions all these equations hold provided that the first few do.

Definition 3. *We call $K \in \mathcal{L}^0$ nonlinear injective if $[K, Q] \in \mathcal{L}^{i+1}$ implies $Q \in \mathcal{L}^{i+1}$ for all $Q \in \mathcal{L}^i$, $i > 0$.*

Definition 4. *We call $K \in \mathcal{L}^0$ relative l-prime with respect to $S \in \mathcal{L}^0$ if $[S, Q] \in \mathrm{Im}\,(\mathrm{ad}(K))$ mod $\mathcal{L}^{i+1}$ implies $Q \in \mathrm{Im}\,(\mathrm{ad}(K))$ mod $\mathcal{L}^{i+1}$ for all $Q \in \mathcal{L}^i$, $i \geqslant l$.*

The following implicit function theorem for filtered Lie algebras, which is to be found in [14], can be used to prove the existence of infinitely many symmetries without the use of extra structures such as a Lax pair, a recursion operator or a master symmetry. The proof is included in appendix A.

Theorem 1 (Sanders, Wang). *Let $\mathcal{L}$ be a filtered Lie algebra. Suppose K, S and $Q \in \mathcal{L}^0$ such that*

- $[K, S] = 0$,
- *K is nonlinear injective,*
- *S is relatively l-prime with respect to K,*
- $[K, Q] \in \mathcal{L}^l$ *and*
- $[S, Q] \in \mathcal{L}^1$;

there exists a unique $\tilde{Q} \in \mathcal{L}^l$ such that $\hat{Q} = Q + \tilde{Q}$ is a symmetry of both K and S, i.e.

- $[K, \hat{Q}] = 0$ *and*
- $[S, \hat{Q}] = 0$.

One has to find infinitely many independent Q for which the conditions are satisfied to prove integrability. This can be done in the symbolic calculus, see [8] and appendix B.

3. A conjecture of Foursov

We put system 4.7 in Jordan form by the invertible linear transformation

$$u_0 \to \tfrac{1}{2}(u_0 + v_0), \qquad v_0 \to \tfrac{1}{2}(u_0 - v_0)$$

then we apply a scale transformation $u_0 \to \frac{1}{2}u_0$ and the parameter translation $\alpha \to \alpha + 2$ to obtain the system we denote by $K(\alpha)$

$$u_t = u_3 + 3u_0u_1$$
$$v_t = \alpha u_1 v_0 + u_0 v_1$$

a generalization of the usual KdV equation. The Foursov conjecture says that for all negative $\alpha \in \mathbb{Q}$ the equation has a hierachy of even-order polynomial symmetries. This is the case, as we show in the following subsections that all conditions of the implicit function theorem are satisfied. Since we allow the symmetries to be nonpolynomial, we find symmetries at any order for any $\alpha \neq 0$.

3.1. $[K, S] = 0$

The first condition in theorem 1 is finding one symmetry (S). Instead of explicitly giving S, we show that for all α the system has infinitely many odd-order symmetries.

Lemma 1. *Let K_n be the (odd) nth-order symmetry of the KdV equation. Then for all n the system*

$$S_n(\alpha) = \begin{pmatrix} K_n \\ (\alpha v_0 + v_1 D_x^{-1})K_{n-2} \end{pmatrix}$$

is a symmetry of $K(\alpha)$.

Proof 1. The bracket

$$D_K S_n(\alpha) - D_{S_n} K(\alpha)$$

has first component $D^u_{K_3} K_n - D^u_{K_n} K_3 = 0$ for K_n is a symmetry of KdV (K_3). The second component is expanded in powers of α. The zeroth power has coefficient

$$\begin{aligned} &v_1 K_n + u_0 v_2 D_x^{-1} K_{n-2} + u_0 v_1 K_{n-2} - v_1 D_x^{-1} D^u_{K_{n-2}} K_3 - D_x^{-1} K_{n-2}(u_0 v_2 + u_1 v_1) \\ &\quad = v_1(K_n + (u_0 - D_x^{-1}(D_x^3 + 3u_0 D_x + 3u_1) - u_1 D_x^{-1})K_{n-2}) \\ &\quad = v_1(K_n - (D_x^2 + 2u_0 + u_1 D_x^{-1})K_{n-2}) \end{aligned}$$

which vanishes because of the recursion relation for KdV symmetries. The coefficient of α

$$v_0(D_x K_n - (D_x^3 + 2u_0 D_x + 3u_1 - u_2 D_x^{-1})K_{n-2})$$

vanishes for the same reason, since $D_x^{-1}u_1 - u_1 D_x^{-1} = D_x^{-1}u_2 D_x^{-1}$. Finally α^2 has coefficient $u_1 v_0 K_{n-2} - u_1 v_0 K_{n-2} = 0$. Therefore the $S_n(\alpha)$ with n odd form a hierarchy of the system $K(\alpha)$ for all α. □

3.2. $K(\alpha)$ is nonlinear injective

As a grading of the Lie algebra $\mathcal{L}$ we choose the degree in u when a system is written as

$$u_t \frac{\partial}{\partial u} + v_t \frac{\partial}{\partial v} = \sum_i K^i.$$

Notice that one can have for example $K^{-1} = v^3 \partial/\partial u$. This grading induces a filtration, $\sum_{i=l} K^i \in \mathcal{L}^l$. For our system $K(\alpha) \in \mathcal{L}^0$ we write

$$K^0 \mod \mathcal{L}^1 = \begin{pmatrix} u_3 \\ 0 \end{pmatrix} \qquad \text{and} \qquad K^1(\alpha) = \begin{pmatrix} 3u_0 u_1 \\ \alpha u_1 v_0 + u_0 v_1 \end{pmatrix}.$$

Lemma 2. *Suppose that $Q \in \mathcal{L}^i$ and nonzero. Then $[K, Q] \equiv 0$ modulo $\mathcal{L}^{i+1}$ implies $i = 0$.*

Proof 2. The first symmetry condition modulo $\mathcal{L}^{i+1}$ reads

$$\begin{aligned} 0 &\equiv [K, Q] \\ &\equiv \begin{pmatrix} D_x^3 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} - \begin{pmatrix} D_{Q_1}^u & D_{Q_1}^v \\ D_{Q_2}^u & D_{Q_2}^v \end{pmatrix} \begin{pmatrix} u_3 \\ 0 \end{pmatrix} \\ &\equiv \begin{pmatrix} D^3 Q_1 - D_{Q_1}^u u_3 \\ D_{Q_2}^u u_3 \end{pmatrix}. \end{aligned}$$

This implies first of all that Q_1 does not contain a part that depends on v because this would be changed by the operation D^3 and left unchanged by $D_{Q_1}^u$. That $Q_1 \in \mathcal{L}^0$ is most easily seen by using the symbolic method (see appendix B). When Q_1 is nonzero

$$(\xi_1 + \xi_2 + \cdots + \xi_{i+1})^3 - (\xi_1^3 + \xi_2^3 + \cdots + \xi_{i+1}^3) = 0$$

only if $i = 0$. Second $[K, Q] \equiv 0$ implies that Q_2 does not depend on u or its derivatives, i.e. $Q_2 \in \mathcal{L}^0$. □

That is to say, $K(\alpha)$ is nonlinear injective.

3.3. S is relatively 2-prime with respect to K

The symmetries we consider in the rest of this paper have the form $(0, Q)$. Suppose now that $Q \in \mathcal{L}^i$. The modulo $\mathcal{L}^{i+1}$ actions of K and S_n are symbolically given by multiplication with the G-functions

$$G_n^i = \xi_1^n + \xi_2^n + \cdots + \xi_i^n.$$

In the symbolic language $[S_n Q] \in \mathrm{Im}\,(\mathrm{ad}(K))$ implies $Q \in \mathrm{Im}\,(\mathrm{ad}(K))$ modulo $\mathcal{L}^{i+1}$ whenever G_3^{i+1} and G_n^{i+1} are relative prime.

Lemma 3. *All G_n^i with $i \geqslant 3$ are irreducible.*

Proof 3. If the projective curve $G_n^3 = 0$ has two components it has a singularity, that is a projective point (ξ_1, ξ_2, ξ_3) where all partial derivatives of G_n^3 vanish. It is easy to see that no such point exists. Thus G_n^3 is irreducible and because $G_n^i = G_n^{i-1}$ at $\xi_i = 0$ all G_n^i with $i > 3$ are irreducible as well. □

This shows that S_n is relatively 2-prime with respect to $K(\alpha)$.

3.4. $[K, Q] \in \mathcal{L}^2$

We look for symmetries of the form $(0, Q_k)$. Then automatically the first equation $KQ_k \equiv 0$ mod $\mathcal{L}^1$ holds (see section 3.2). The next (already the last) equation is written modulo $\mathcal{L}^2$

$$0 \equiv \begin{pmatrix} D_x^3 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ Q_k^1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ D^u_{Q_k^1} & D^v_{Q_k^1} \end{pmatrix}\begin{pmatrix} u_3 \\ 0 \end{pmatrix} + \begin{pmatrix} 3(u_1 + u_0 D_x) & 0 \\ \alpha v_0 D_x + v_1 & \alpha U_1 + u_0 D_x \end{pmatrix}\begin{pmatrix} 0 \\ Q_k^0 \end{pmatrix}$$
$$- \begin{pmatrix} 0 & 0 \\ D^u_{Q_k^0} & D^v_{Q_k^0} \end{pmatrix}\begin{pmatrix} 3u_0u_1 \\ \alpha u_1 v_0 + u_0 v_1 \end{pmatrix}$$

leading to

$$D^u_{Q_k^1} u_3 \equiv u_0 D_x Q_k^0 + \alpha u_1 Q_k^0 - D^v_{Q_k^0}(\alpha u_1 v_0 + u_0 v_1)$$

which can be solved if the coefficients of u_0, u_1 and u_2 vanish. Expanding the right-hand-side terms gives

$$\begin{aligned} u_0 D_x Q_k^0 + \alpha u_1 Q_k^0 - D^v_{Q_k^0}(\alpha u_1 v_0 + u_0 v_1) &\equiv u_0(D_x Q_k^0 - v_{i+1}\partial_{v_i} Q_k^0) \\ &+ u_1(\alpha Q_k^0 - (\alpha + i) v_i \partial_{v_i} Q_k^0) \\ &+ u_2\left(-\alpha i - \frac{i(i-1)}{2}\right) v_{i-1}\partial_{v_i} Q_k^0 + \cdots \end{aligned}$$

where the sum over i is taken. Since total differentiation is performed by the operator $D_x = v_{i+1}\partial_{v_i}$ the coefficient of u_0 vanishes identically.

Let $\alpha \neq 0$. We make the following ansatz.

Ansatz 1. *The term of lowest grading has the form*

$$Q_k^0 \equiv \sum_{j=0}^{2k} c_j v_j v_{2k-j} v_0^{w/2-k-1} \text{ modulo } \mathcal{L}^1$$

of order $2k$ *and weight* w. *Here* k *is a positive integer and* w *can be any number.*

The operator $i v_i \partial_{v_i}$ counts the order, it multiplies Q_k^0 with $2k$. The operator $v_i \partial_{v_i}$ counts the degree in v; it multiplies Q_k^0 by $(w/2) - k + 1$. Therefore the u_1-coefficient vanishes when

$$w = 2k\frac{\alpha - 2}{\alpha}.$$

When we put $w = 2k + 2n$ we obtain $\alpha + 2 = 2(1 - (k/n))$ as predicted by Foursov in his conjecture. If $n \in \mathbb{N}$ this is where the symmetries are polynomial.

Straightforward calculation shows that the vanishing of the u_2-coefficient implies

$$c_j = c_{j-1}\frac{(j - 1 - 2k)(2\alpha + 2k - j)}{j(2\alpha + j - 1)}.$$

As long as $\alpha \neq 0, -1/2, \ldots, 1/2 - k$ we can solve this recursion relation.

The result is nonempty because $c_{k+i} = c_{k-i}$ when $k \in \mathbb{N}$, which can be easily proven by induction on i. One can look for odd-order solutions; take for k a half integer. In this case we have $c_{k+1/2+i} = -c_{k-1/2-i}$, which implies $Q_k^0 = 0$. However when $-2\alpha \in \mathbb{N}$ and $0 < 2k + 2\alpha \leqslant k$ we have $c_j = 0$ for all $j \geqslant 2k + 2\alpha$. This means that when $-\alpha$ is integer or half integer there exist respectively $-\alpha$ and $-2(\alpha + 1)$ additional odd-order solutions.

Example 1. The only additional odd-order symmetry with this form of $K\left(-3/2\right)$ is

$$
\begin{aligned}
u_t &= 0\\
v_t &= v_0v_5 + \tfrac{5}{3}v_4v_1 + \tfrac{25}{3}u_1v_1^2 + \tfrac{25}{3}u_0v_1v_2 + 10u_1v_0v_2 + 5u_0v_3v_0\\
&\quad + 9u_2v_0v_1 + \tfrac{3}{2}u_3v_0^2 + \tfrac{9}{2}u_0u_1v_0^2 + 6u_0^2v_1v_0.
\end{aligned}
$$

To cover the higher values of k for integer or half-integer negative α we start counting coefficients from the other side of the polynomial. The assumption we must make here is that $k \leqslant -\alpha$ or $k > -2\alpha$ whenever $-2\alpha \in \mathbb{N}$.

Ansatz 2. *Let*

$$
Q_k^0 \equiv \sum_{i=0}^{k} b_i v_{k+i} v_{k-i} v_0^{w/2-k-1} \text{ modulo } \mathcal{L}^1.
$$

Then the recurrence becomes

$$
\begin{aligned}
b_1 &= 2b_0 \frac{k(1-k-2\alpha)}{(k+1)(2\alpha+k)}\\
b_i &= b_{i-1} \frac{(k+1-i)(i-k-2\alpha)}{(k+i)(k+i-1+2\alpha)}.
\end{aligned}
$$

When $-2\alpha \in \mathbb{N}$ and $k = -2\alpha + 1 + i$, $i \in \mathbb{N}$ all coefficients b_j, $j > i$ vanish.

It is possible to perform the computations in higher filtration spaces. A recursive formula in symbolic language for the terms Q_k^n modulo $\mathcal{L}^{n+1}$ is given in appendix C. There, MAPLE (see [4]) computer code that produces these kinds of symmetry and an explicit example with complex α is presented as well.

There is more symmetry. We make another ansatz.

Ansatz 3. *Let*

$$
Q_k^0 \equiv \sum_{j=0}^{k} a_j v_{k-j} v_1^j v_0^{w/2-k/2-j} \text{ modulo } \mathcal{L}^1
$$

of order k and weight w; again k is a positive integer and $w \in \mathbb{C}$.

The coefficient of u_1 vanishes if $w = k\frac{\alpha-2}{\alpha}$ and the coefficient of u_2 vanishes if

$$
a_{j+1} = \frac{a_j(k-j)(j+1-2\alpha-k)}{2\alpha(j+1)}.
$$

This procedure works for all integer $k > 1$ and all $w \in \mathbb{C}$. We have $Q_k^0 = 0$ when $k = 1$. For $k = 2$ one obtains the same symmetries as taking $k = 1$ in ansatz 1 (or 2). When α is a negative integer or half integer we observe that $a_j = 0$ for all $j > k - 1 + 2\alpha$.

Example 2. $K\left(-4/3\right)$ has the extra symmetry of order four and weight ten

$$
\begin{aligned}
u_t &= 0\\
v_t &= v_4v_0^3 + \tfrac{1}{2}v_0^2v_3v_1 - \tfrac{3}{16}v_0v_2v_1^2 + \tfrac{15}{256}v_1^4 + \tfrac{4}{3}u_2v_0^4\\
&\quad + 5u_1v_0^3v_1 + 4u_0v_0^3v_2 + \tfrac{5}{4}u_0v_0^2v_1^2 + \tfrac{4}{3}u_0^2v_0^4.
\end{aligned}
$$

3.5. $[S, Q] \in \mathcal{L}^1$

The first component of S_n does not depend on v and its second vanishes modulo $\mathcal{L}^1$. Moreover the first component of Q_k vanishes and its second does not depend on u modulo $\mathcal{L}^1$. These properties make their bracket vanish modulo $\mathcal{L}^1$.

4. Results

We have shown that the KdV equation coupled to a nonlinear equation

$$K(\alpha): \begin{array}{l} u_t = u_3 + 3u_0u_1 \\ v_t = \alpha u_1 v_0 + u_0 v_1 \end{array}$$

has infinitely many odd-order symmetries $S_n(\alpha)$ and that its linear part is nonlinear injective. The linear part of any odd-order symmetry $S_n(\alpha)$ is relatively 2-prime with $K(\alpha)$. We solved the first two symmetry conditions $[K, Q_k] \in \mathcal{L}^2$ for infinitely many Q_k (twice) for all α and showed that $[S_n, Q_k] \in \mathcal{L}^1$. By the implicit function theorem there exist $\hat{Q}_k(\alpha)$ which commute with $K(\alpha)$ and with all $S_n(\alpha)$.

There is a linear map that transforms every symmetry of $K(\alpha)$ into a symmetry of system 4.7 found by Foursov. His conjecture turns out to be true inside the class of polynomial symmetries. However, the symmetry structure of the equation is bigger than this.

Appendix A. Implicit function theorem

Lemma 4. *Let $\mathcal{L}$ be a filtered Lie algebra. Suppose K, S and $Q \in \mathcal{L}$ such that*

- $[K, S] = 0$,
- *K is nonlinear injective,*
- $[K, Q] \in \mathcal{L}^l$ *and*
- $[S, Q] \in \mathcal{L}^1$.

Then

- $[S, Q] \in \mathcal{L}^l$.

Proof 4. We know $[K, [S, Q]] = [S, [K, Q]] \in \mathcal{L}^l$. Because $[S, Q] \in \mathcal{L}^1$ we can use the nonlinear injectiveness of K to conclude that $[S, Q] \in \mathcal{L}^l$. □

Theorem 2 (Sanders, Wang). *Under the conditions in lemma 4 and the additional condition*

- *S is relatively l-prime with respect to K*

there exists a unique $\tilde{Q} \in \mathcal{L}^l$ such that $\hat{Q} = Q + \tilde{Q}$ is an invariant of both K and S, i.e.

- $[K, \hat{Q}] = 0$ *and*
- $[S, \hat{Q}] = 0$.

Proof 5. By induction we show that there exists a $\hat{Q}$ such that $[K, \hat{Q}] \in \mathcal{L}^p$ and $[S, \hat{Q}] \in \mathcal{L}^p$ for all $p \geqslant l$. Suppose $[K, Q] \in \mathcal{L}^p$ and $[S, Q] \in \mathcal{L}^p$ hold for some $p \geqslant l$. The case $p = l$ follows from lemma 4. We have

$$[K, [S, Q]] = [S, [K, Q]]$$

and, in particular, $[S, [K, Q]] \in \mathrm{Im}(\mathrm{ad}(K)) \bmod \mathcal{L}^{p+1}$. By the relative l-primeness of S with respect to K we have that $[K, Q] \in \mathrm{Im}(\mathrm{ad}(K)) \bmod \mathcal{L}^{p+1}$. Therefore we can uniquely define $\tilde{Q} \in \mathcal{L}^p$ by

$$[K, \tilde{Q}] = -[K, Q]$$

such that $\hat{Q} = Q + \tilde{Q}$ satisfies $[K, \hat{Q}] \in \mathcal{L}^{p+1}$ and by lemma 4 (taking $l = p+1$) $[S, \hat{Q}] \in \mathcal{L}^{p+1}$.

This implies that Q can always be extended such that all homogeneous parts of $[K, Q]$ and $[S, Q]$ vanish. Uniqueness follows from the assumption that $\cap_{i=0}^{\infty}\mathcal{L}^i = \{0\}$. □

Appendix B. Symbolic calculus

The Gel'fand-Dikiĭ transformation, cf [8], is a one to one mapping between differential polynomials and symmetric polynomials. The basic idea is very old, probably dating from the time when the position of index and power were not as fixed as they are today. We give some rules without proof.

A differential monomial with m symbols of the form u_k

$$M(u) = \prod_{j=1}^{m} u_{i_j}$$

is mapped to

$$M(\xi) = \frac{1}{m!} \sum_{\sigma_m} \prod_{j=1}^{m} \xi_j^{i_j}$$

where $\sum_{\sigma_m}$ means one has to sum over all different permutations of the integers $1, \ldots, m$. Monomials act on each other as follows: let N have n ξ-symbols

$$M(\xi) \circ N(\xi) = \frac{1}{(m+n)!} \sum_{\sigma_{m+n}} M(\xi) N(\xi).$$

This mapping is extended to differential monomials in more variables by introducing other symbols

$$M(u)N(v) \to M(\xi)N(\zeta).$$

One symmetrizes only in the symbols with the same name since $u_i u_j = u_j u_i$ and $u_i v_j \neq u_j v_i$.

The operation of taking a total derivative turns into multiplication with the sum of all symbols involved. Let K have m ξ-symbols and n ζ-symbols

$$D_x K(u, v) \to \left(\sum_{i=1}^{m} \xi_i + \sum_{j=1}^{n} \zeta_j \right) K(\xi, \zeta).$$

Taking the Frechet derivative of a differential polynomial is done as follows:

$$D^u_{M(u)} = \sum_{k=1}^{m} \left(\prod_{j=1, j \neq k}^{m} u_{i_j} \right) D_x^{i_k}$$

and in the symbolic calculus, when there are other symbols involved as well,

$$D^u_{K(\xi,\zeta)} = n K(\xi_1, \ldots, \xi_{n-1}, D, \zeta) \circ$$

where ξ_n is replaced by the symbol D which is representing the sum of all symbols in the monomial the Frechet derivative is acting on.

Appendix C. Higher-order calculations

Symmetries of $K(\alpha)$ are symbolically given by

$$Q(\alpha, k) = \begin{pmatrix} 0 \\ \sum_{n=0}^{k} Q^n \end{pmatrix}$$

where Q^0 is given by function $F[k, \alpha](\zeta_1, \zeta_2) v_0^{-(2k+a/a)}$ where F satisfies the linear differential equation

$$\alpha(\partial_{\zeta_1} + \partial_{\zeta_2})F + \tfrac{1}{2}(\zeta_1 \partial^2_{\zeta_1} + \zeta_2 \partial^2_{\zeta_2})F = 0.$$

The higher-order Q^i satisfy the recurrence relation

$$\left(n\sum_{i=1}^{n}\xi_i^3\right)Q^n = \sum_{j=1}^{n}\left(\sum_{i=1,i\neq j}^{n}(\xi_i)+2(\alpha+k)\xi_j+\zeta_1+\zeta_2\right)Q^{n-1}(\xi_{n/j},\zeta_1,\zeta_2)$$
$$-(\alpha\xi_j+\zeta_1)Q^{n-1}(\xi_{n/j},\zeta_2,\xi_j+\zeta_1)-(\alpha\xi_j+\zeta_2)Q^{n-1}(\xi_{n/j},\zeta_1,\xi_j+\zeta_2)$$
$$-3\sum_{i=1}^{n}\sum_{k>i}^{n}(\xi_i+\xi_k)Q^{n-1}(\xi_{n/i/k},\xi_i+\xi_k,\zeta_1,\zeta_2)$$

where $\xi_{n/i} = \xi_1,\ldots,\xi_{i-1},\xi_{i+1},\ldots,\xi_n$. The implicit function theorem guarantees that this relation generates polynomials, which can be transformed into differential functions. This transformation is done in MAPLE (see [4]) by the following function `TRANS(P,n)`, which transforms polynomials $P(x_1,\ldots,x_n,y_1,y_2)$ into the corresponding differential polynomial with degree n in u and two in v.

```
TRANS:=proc(P,n)
local R,e,i,Q:
R:=0:
Q:=expand(P):
if type(Q,'+') then Q:=convert(Q,list) else Q:=[Q] fi:
for e in Q do
for i to n do e:=e*u[degree(e,x[i])]/x[i]^degree(e,x[i]) od:
for i to 2 do e:=e*v[degree(e,y[i])]/y[i]^degree(e,y[i]) od:
R:=R+e od:
RETURN(R)
end:
```

The symmetries can be calculated on a computer in the following way. First set (`a` is the same as α)

```
a:=-4/3:
k:=2:
```

then run the program

```
c[0]:=1/2:
if type(2*a,integer) and a<0 and k>-2*a then
F:=c[0]*(y[1]*y[2])^k:
for i to k+2*a-1 do
c[i]:=-c[i-1]*(k+1-i)*(k+2*a-i)/(k+i)/(k+i-1+2*a):
F:=F+c[i]*(y[1]^(k+i)*y[2]^(k-i)+y[2]^(k+i)*y[1]^(k-i)) od:
else
F:=c[0]*(y[1]^(2*k)+y[2]^(2*k)):
for i to k-1 do
c[i]:=c[i-1]*(i-1-2*k)*(2*k+2*a-i)/i/(i-1+2*a):
F:=F+c[i]*(y[1]^i*y[2]^(2*k-i)+y[2]^i*y[1]^(2*k-i)) od:
F:=F-c[k-1]*(k+1)*(k+2*a)/k/(k-1+2*a)*y[1]^k*y[2]^k fi:
Q:=TRANS(F,0):
F:=unapply(F,y[1],y[2]):
for n to k do G:=0:
for j to n do
G:=G+(sum(x['i'],'i'=1..n)+(2*a+2*k-1)*x[j]+y[1]+y[2])
*F(seq(x[i],'i'=1..j-1),seq(x[i],'i'=j+1..n),y[1],y[2])
```

```
-(a*x[j]+y[1])*F(seq(x[i],`i`=1..j-1),seq(x[i],`i`=j+1..n)
,y[2],x[j]+y[1])-(a*x[j]+y[2])*F(seq(x[i],`i`=1..j-1)
,seq(x[i],`i`=j+1..n),y[1],x[j]+y[2]):
for l from j+1 to n do
G:=G-3*(x[j]+x[l])*F(seq(x[i],`i`=1..j-1),seq(x[i],`i`
=j+1..l-1),seq(x[i],`i`=l+1..n),x[j]+x[l],y[1],y[2]) od od:
G:=factor(G/sum(x['i']^3,'i'=1..n)/n):
Q:=Q+TRANS(G,n):
F:=unapply(G,seq(x[i],`i`=1..n),y[1],y[2]) od:
Q:=[0,factor(Q)*v[0]^(factor(-(2*k+a)/a))];
```

to find the second symmetry of K when $\alpha = -4/3$, it has the same order and weight as example 2 in section 3.4. The whole procedure also works for complex α.

Example 3. When one sets

```
k:=1:
alias(a=RootOf(x^2+x+1,x)):
```

the program calculates the first symmetry

```
Q := [0, 1/6*(-1+a)*(-4*v[2]*v[0]+3*v[1]^2+2*a*u[0]*v[0]^2
                -2*a*v[2]*v[0]-2*v[0]^2*u[0])*v[0]^(1+2*a)].
```

It can easily be checked that this Q commutes with

```
K := [u[3]+3*u[1]*u[0], a*u[1]*v[0]+u[0]*v[1]];
```

for primitive third roots of unity a.

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