

Research



CrossMark
click for updates

Cite this article: Hone ANW, van der Kamp PH, Quispel GRW, Tran DT. 2013 Integrability of reductions of the discrete Korteweg–de Vries and potential Korteweg–de Vries equations. *Proc R Soc A* 469: 20120747. <http://dx.doi.org/10.1098/rspa.2012.0747>

Received: 25 January 2013

Accepted: 13 March 2013

Subject Areas:

mathematical physics

Keywords:

partial difference equations, integrable maps, Poisson brackets

Author for correspondence:

A. N. W. Hone

e-mail: a.n.w.hone@kent.ac.uk

Electronic supplementary material is available at <http://dx.doi.org/10.1098/rspa.2012.0747> or via <http://rspa.royalsocietypublishing.org>.

Integrability of reductions of the discrete Korteweg–de Vries and potential Korteweg–de Vries equations

A. N. W. Hone¹, P. H. van der Kamp², G. R. W. Quispel² and D. T. Tran³

¹School of Mathematics, Statistics and Actuarial Science, University of Kent, Canterbury, UK

²Department of Mathematics and Statistics, La Trobe University, Melbourne, Victoria, Australia

³School of Mathematics and Statistics, University of New South Wales, Sydney, New South Wales, Australia

We study the integrability of mappings obtained as reductions of the discrete Korteweg–de Vries (KdV) equation and of two copies of the discrete potential KdV (pKdV) equation. We show that the mappings corresponding to the discrete KdV equation, which can be derived from the latter, are completely integrable in the Liouville–Arnold sense. The mappings associated with two copies of the pKdV equation are also shown to be integrable.

1. Introduction

The problem of integrating differential equations goes back to the origins of calculus and its application to problems in classical mechanics. In the nineteenth century, the notion of complete integrability was provided with a solid theoretical foundation by Liouville, whose theorem gave sufficient conditions for a Hamiltonian system to be integrated by quadratures; yet only a few examples of integrable mechanical systems (mostly with a small number of degrees of freedom) were known at the time. Poincaré’s subsequent results on the non-integrability of the three-body problem seemed to indicate that many, if not most, systems should be non-integrable. Nevertheless, examples of integrable systems (and action–angle variables in particular) played an important role in the early development of quantum theory.

The theory of integrable systems only began to expand rapidly in the latter part of the twentieth century, with the discovery of the remarkable properties of the Korteweg–de Vries (KdV) equation, together with a host of other nonlinear partial differential equations that were found to be amenable to the inverse scattering technique. As well as having exact pulse-like solutions (solitons) that undergo elastic collisions, such equations could be interpreted as infinite-dimensional Hamiltonian systems, with an infinite number of conserved quantities. Moreover, it was shown that these equations admit particular reductions (e.g. to stationary solutions, or to travelling waves) that can be viewed as integrable mechanical systems with finitely many degrees of freedom. The papers in the collection [1] provide a concise and self-contained survey of the theory of integrable ordinary and partial differential equations; for a more recent set of review papers, see [2].

In the past two decades or so, there has been a more gradual development of *discrete* integrable systems, in the form of finite-dimensional maps [3,4] and discrete Painlevé equations [5], as well as partial difference equations defined on lattices or quad-graphs [6,7]. Discrete integrable systems can be obtained directly by seeking discrete analogues of particular continuous soliton equations or Hamiltonian flows [8], but they also appear independently in solvable models of statistical mechanics (see the link with the hard hexagon model in Quispel *et al.* [3]) or quantum field theory [9]. An important theoretical result for ordinary difference equations or maps is the fact that the Liouville–Arnold definition of integrability for Hamiltonian systems of ordinary differential equations can be extended naturally to symplectic maps [10–12], so that an appropriate modification of Liouville’s theorem holds. For lattice equations with two or three independent variables, there is less theory available (especially from the Hamiltonian point of view), and the full details of the known integrable examples are still being explored, but one way to gain understanding is through the analysis of particular families of reductions.

By imposing a periodicity condition, integrable lattice equations can be reduced to ordinary difference equations (or mappings/maps) [4,13–15]. It is believed that the reduced maps obtained from an *integrable* lattice equation are completely integrable in the Liouville–Arnold sense. To *prove* that a map is integrable, one needs to find a Poisson structure together with a sufficient number of functionally independent first integrals, and then show that these integrals commute with respect to the Poisson bracket. One complication that immediately arises is that the reduced maps naturally come in families of increasing dimension, and the number of first integrals grows with the dimension. The complete integrability of some particular KdV-type maps was proved in Capel *et al.* [4], and progress has been made recently with other families of maps. For maps obtained as reductions of the equations in the Adler–Bobenko–Suris (ABS) classification [7], and for reductions of the sine-Gordon and modified Korteweg–de Vries (mKdV) equations, first integrals were given in closed form by using the staircase method and the non-commutative Vieta expansion [16,17]. In particular, the complete integrability of mappings obtained as reductions of the discrete sine-Gordon, mKdV and potential KdV (pKdV) equations was studied in detail by Tran and co-workers [18,19].

Given a map, the question arises as to whether it has a Poisson structure, and if so, how can one find it? In general, the answer is not known. However, for some classes of maps, one can assume that in coordinates x_j the Poisson structure is in canonical or log-canonical form, i.e. the Poisson brackets have the form $\{x_i, x_j\} = \Omega_{ij}$ or $\{x_i, x_j\} = \Omega_{ij} x_i x_j$, respectively, where Ω is a constant skew-symmetric matrix. This approach is effective for mappings obtained in the context of cluster algebras [20–22], and also applies to reductions of the lattice pKdV, sine-Gordon and mKdV equations [19]. Another approach requires the existence of a Lagrangian for the reduced map: by using a discrete analogue of the Ostrogradsky transformation, as introduced in Bruschi *et al.* [11], one can rewrite the map in canonical coordinates; from there, one can derive a Poisson structure in the original variables.

Here, we start by considering a well-known integrable lattice equation, namely the discrete pKdV equation, also referred to as H_1 in the ABS list [7], which is given by

$$(u_{\ell,m} - u_{\ell+1,m+1})(u_{\ell+1,m} - u_{\ell,m+1}) = 1, \quad (1.1)$$

where $(\ell, m) \in \mathbb{Z}^2$. Early results on this equation appear in earlier studies [4,6,23,24], where among other things, it was shown that (1.1) leads to the continuous pKdV equation, that is

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} + 3 \left(\frac{\partial u}{\partial x} \right)^2, \quad (1.2)$$

by performing a suitable continuum limit. In Capel *et al.* [4], a Lagrangian was obtained for the discrete pKdV equation (1.1); this can be explained using the so-called three-leg form, as in Adler *et al.* [7]. However, the associated Euler–Lagrange equation turns out to consist of two copies of the lattice pKdV equation, that is

$$u_{\ell+1,m} + u_{\ell-1,m} + \frac{1}{u_{\ell-1,m-1} - u_{\ell,m}} = u_{\ell,m+1} + u_{\ell,m-1} + \frac{1}{u_{\ell,m} - u_{\ell+1,m+1}}. \quad (1.3)$$

The latter equation, which henceforth we refer to as the *double pKdV equation*, is somewhat more general than (1.1): every solution of (1.1) is a solution of (1.3), but the converse statement does not hold. In this paper, we shall be concerned with the double pKdV equation (1.3), rather than with (1.1).

Next, we introduce a new variable on the lattice, $v_{\ell,m} := u_{\ell,m} - u_{\ell+1,m+1}$, and immediately find that, whenever $u_{\ell,m}$ is a solution of (1.3), $v_{\ell,m}$ satisfies

$$v_{\ell+1,m} - v_{\ell,m+1} = \frac{1}{v_{\ell,m}} - \frac{1}{v_{\ell+1,m+1}}. \quad (1.4)$$

The latter equation is known as the lattice KdV equation. Both (1.1) and (1.4) are integrable lattice equations, in the sense that they can be derived as the compatibility condition for an associated linear system, known as a Lax pair; this is discussed in §4.

In this paper, we perform the so-called $(d-1, -1)$ -reduction on the discrete pKdV Lagrangian and derive the corresponding reduction of the double pKdV equation (1.3). This means that we consider functions $u = u_{\ell,m}$ on the lattice which have the following periodicity property under shifts:

$$u_{\ell,m} = u_{\ell+d-1,m-1}. \quad (1.5)$$

Such periodicity implies that u depends on the lattice variables ℓ and m through the combination $n = \ell + m(d-1)$ only; thus, with a slight abuse of notation, we write $u = u_n$. Such a reduction can be understood as a discrete analogue of the travelling wave reduction of a partial differential equation: for a function $u(x, t)$ satisfying a suitable partial differential equation such as (1.2), one considers solutions that are invariant under $x \rightarrow x + c\delta$, $t \rightarrow t + \delta$ for all δ ; such solutions depend on x, t through the combination $z = x - ct$ only, corresponding to waves travelling with constant speed c . By analogy with the continuous case, where one obtains ordinary differential equations (with independent variable z) for the travelling waves, it is apparent that imposing the condition (1.5) in the discrete setting leads to ordinary difference equations (with independent variable n). Note that in the continuous case this reduction yields a one-parameter family of ordinary differential equations of fixed order (with parameter c), whereas in the discrete case, one finds a family of ordinary difference equations whose order depends on d .

Here, we are concerned with the complete integrability of the ordinary difference equation obtained as the $(d-1, -1)$ -reduction of the double pKdV equation (1.3), which is equivalent to a birational map in dimension $2d$, and the associated reduction of the lattice KdV equation (1.4), which gives a map in dimension d . We begin by finding Poisson brackets for reductions of the double pKdV equation, which are then used to infer brackets for the corresponding maps obtained from lattice KdV; having found first integrals and proved Liouville integrability for the KdV maps, we are subsequently able to do the same for the double pKdV maps.

The paper is organized as follows. In §2, we start with the discrete Lagrangian whose Euler–Lagrange equation is the double pKdV equation (1.3). For each d , we then derive a symplectic structure for the double pKdV map obtained as the $(d-1, -1)$ -reduction of this discrete Euler–Lagrange equation, by using a discrete analogue of the Ostrogradsky transformation. This provides a non-degenerate Poisson bracket for each of these maps. In §3, we present a

Poisson structure for the associated map obtained as the $(d-1, -1)$ -reduction of the lattice KdV equation (1.4), which is induced from the bracket found in §2. The Hirota bilinear form of each of the KdV maps is also given, in terms of tau-functions, whence (via a link with cluster algebras) we derive a second Poisson structure that is compatible with the first. In §4, we present closed-form expressions for first integrals of each reduced KdV map, and we show that they are in involution with respect to the first of the Poisson structures. This furnishes a direct proof of Liouville integrability for these KdV maps, within the framework of the papers [16,19]. Another proof, based on the pair of compatible Poisson brackets, is also sketched. In §5, we return to the double pKdV maps, and present first integrals for each of them. Some of these integrals are derived from the integrals of the corresponding KdV map in the previous section, whereas additional commuting integrals are found using a function periodic with period d ; this is a d -integral, in the sense of Haggard *et al.* [25]. The paper ends with some brief conclusions, and some additional comments are relegated to electronic supplementary material, appendix.

2. Poisson structure for the double potential Korteweg–de Vries maps

Henceforth, it is convenient to use $\tilde{\cdot}$ and $\hat{\cdot}$ to denote shifts in ℓ and m directions, respectively, so that $\tilde{u} = u_{\ell+1,m}$, $\hat{u} = u_{\ell,m+1}$, etc. It is known that all equations of the form

$$Q(u, \tilde{u}, \hat{u}, \hat{\tilde{u}}, \alpha, \beta) = 0 \quad (2.1)$$

in the ABS list [7] are equivalent (up to some transformations) to the existence of an equation of the so-called three-leg type, that is

$$P(u, \tilde{u}, \hat{u}, \hat{\tilde{u}}; \alpha, \beta) \equiv \phi(u, \hat{u}, \alpha) - \phi(u, \tilde{u}, \beta) - \psi(u, \hat{\tilde{u}}, \alpha, \beta) = 0, \quad (2.2)$$

for suitable functions ϕ, ψ , where α and β are parameters; the latter leads to the derivation of a Lagrangian for each of the equations (2.1). In particular, for the pKdV equation (1.1), which is a (parameter-free) equation of the form (2.1), we have $\phi(u, \tilde{u}) = u + \tilde{u}$ and $\psi(u, \hat{\tilde{u}}) = 1/(u - \hat{\tilde{u}})$, and using the three-leg form (2.2) leads to the Lagrangian

$$\mathcal{L} = \frac{1}{2}(u + \tilde{u})^2 - \frac{1}{2}(u + \hat{u})^2 - \log |u - \hat{\tilde{u}}|. \quad (2.3)$$

The corresponding discrete Euler–Lagrange equation is the double pKdV equation (1.3), which can be rewritten as

$$\hat{\tilde{J}} - J = 0, \quad \text{with } J = \tilde{u} - \hat{u} - \frac{1}{(u - \hat{\tilde{u}})}. \quad (2.4)$$

The latter equation is more general than (1.1), which arises in the special case that J is identically zero.¹

Now setting $n = \ell + m(d-1)$, with u satisfying (1.5), the $(d-1, -1)$ -reduction applied to the Lagrangian in (2.3) gives $\mathcal{L} = \mathcal{L}(u_n, u_{n+1}, \dots, u_{n+d})$, where

$$\mathcal{L} = \frac{1}{2}(u_n + u_{n+1})^2 - \frac{1}{2}(u_n + u_{n+d-1})^2 - \log |u_n - u_{n+d}|. \quad (2.5)$$

The discrete action functional is $S := \sum_{n \in \mathbb{Z}} \mathcal{L}(u_n, u_{n+1}, \dots, u_{n+d})$. It yields the discrete Euler–Lagrange equation

$$\frac{\delta S}{\delta u_n} = \sum_{r=0}^d \frac{\partial \mathcal{L}(u_{n-r}, u_{n+1-r}, \dots, u_{n+d-r})}{\partial u_n} = \sum_{r=0}^d \mathcal{E}^{-r} \mathcal{L}_r = 0, \quad (2.6)$$

¹However, every solution of (1.3) can be written as $u = U + a$, where U is a solution of (1.1), and a is a solution of the linear equation $\hat{\tilde{a}} = a$. See the electronic supplementary material (appendix) for more details.

where \mathcal{E} denotes the shift operator and $\mathcal{L}_r = \partial \mathcal{L}(u_n, u_{n+1}, \dots, u_{n+d}) / \partial u_{n+r}$. Thus, we obtain the ordinary difference equation

$$u_{n+1} - u_{n+d-1} + u_{n-1} - u_{n-d+1} - \frac{1}{u_n - u_{n+d}} + \frac{1}{u_{n-d} - u_n} = 0, \quad (2.7)$$

which is precisely the $(d-1, -1)$ -reduction of (1.3). The solutions of this equation are equivalent to the iterates of the $2d$ -dimensional map

$$(u_{n-d}, u_{n-d+1}, \dots, u_{n+d-1}) \mapsto (u_{n-d+1}, u_{n-d+2}, \dots, u_{n+d}), \quad (2.8)$$

where u_{n+d} is found from equation (2.7).

Given a Lagrangian of first order for a classical mechanical system, the Legendre transformation produces canonical symplectic coordinates on the phase space; the Ostrogradsky transformation is the analogue of this for Lagrangians of higher order [26]. In order to derive a non-degenerate Poisson bracket for the $2d$ -dimensional map, we use a discrete analogue of the Ostrogradsky transformation, as given in Bruschi *et al.* [11], which is a change of variables to canonical coordinates, $(u_{n-d}, u_{n-d+1}, \dots, u_{n+d-1}) \rightarrow (q_1, \dots, q_d, p_1, \dots, p_d)$, where $q_i = u_{n+i-1}$, $p_i = \mathcal{E}^{-1} \sum_{r=0}^{d-i} \mathcal{E}^{-r} \mathcal{L}_{r+i}$. Thus, from (2.5), we obtain

$$\begin{aligned} q_i &= u_{n+i-1}, \quad i = 1, \dots, d; \\ p_1 &= -u_{n+1} + u_{n+d-1} + \frac{1}{u_n - u_{n+d}} = u_{n-1} - u_{n+1-d} + \frac{1}{u_{n-d} - u_n}; \\ p_i &= -u_{n+i-d} - u_{n+i-1} + \frac{1}{u_{n-d+i-1} - u_{n+i-1}}, \quad i = 2, \dots, d-1; \end{aligned}$$

and

$$p_d = \frac{1}{u_{n-1} - u_{n+d-1}}.$$

In terms of the canonical coordinates (q_j, p_j) , the map (2.7) is rewritten as

$$\begin{aligned} q_i &\mapsto q_{i+1}, \quad i = 1, \dots, d-1; \quad q_d \mapsto q_1 - (p_1 + q_2 - q_d)^{-1}; \\ p_1 &\mapsto p_2 + q_2 + q_1; \quad p_i \mapsto p_{i+1}, \quad i = 2, \dots, d-2; \end{aligned}$$

and

$$p_{d-1} \mapsto p_d - q_1 - q_d; \quad p_d \mapsto p_1 + q_2 - q_d.$$

By the general results in Bruschi *et al.* [11], this map is symplectic with respect to the canonical symplectic form $\sum_{j=1}^d dp_j \wedge dq_j$. Equivalently, it preserves the canonical Poisson brackets $\{p_i, p_j\} = \{q_i, q_j\} = 0$, $\{p_i, q_j\} = \delta_{ij}$.

In order to find the Poisson brackets for the coordinates u_n , we write them in terms of (q_j, p_j) . For all $0 \leq i \leq d-1$, we have $u_{n+i} = q_{i+1}$, and

$$u_{n-1} = q_d + \frac{1}{p_d} = [q_d; p_d],$$

$$u_{n-i-1} = [q_{d-i}; p_{d-i} + q_{d-i} + u_{n-i}] \quad (\text{for } 1 \leq i \leq d-2)$$

and

$$u_{n-d} = [q_1; p_1 + u_{n+1-d} - u_{n-1}],$$

which means that for $1 \leq i \leq d-2$, u_{n-i-1} is found recursively as

$$[q_{d-i}; p_{d-i} + q_{d-i} + q_{d-i+1}, p_{d-i+1} + q_{d-i+1} + q_{d-i+2}, \dots, p_{d-1} + q_{d-1} + q_d, p_d],$$

while $u_{n-d} = [q_1; p_1 - q_d + q_2 - 1/p_d, p_2 + q_2 + q_3, \dots, p_{d-1} + q_{d-1} + q_d, p_d]$, where $[\ ;]$ denotes a continued fraction. This yields the following results.

Theorem 2.1. *The two-dimensional map given by (2.8) with (2.7) is a Poisson map with respect to the non-degenerate bracket given for $0 \leq j < d - 1$ by*

$$\text{and } \left. \begin{aligned} \{u_{n-d}, u_{n-j-1}\} &= 0, \\ \{u_{n-d}, u_{n+j}\} &= (-1)^{j+1} (u_{n-d} - u_n)^2 \dots (u_{n-d+j} - u_{n+j})^2 \\ \{u_{n-d}, u_{n+d-1}\} &= (-1)^d (u_{n-d} - u_n)^2 \dots (u_{n-1} - u_{n+d-1})^2 \\ &\quad - (u_{n-d} - u_n)^2 (u_{n-1} - u_{n+d-1})^2. \end{aligned} \right\} \quad (2.9)$$

Proof. In order to prove the first of the formulae in (2.9), we note that

$$\begin{aligned} \{u_{n-d}, u_{n-r-1}\} &= \left\{ \left[q_1; p_1 - q_d + q_2 - \frac{1}{p_d}, p_2 + q_2 + q_3, \dots, p_{d-1} + q_{d-1} + q_d, p_d \right], \right. \\ &\quad \left. [q_{d-r}; p_{d-r} + q_{d-r} + q_{d-r+1}, \dots, p_{d-1} + q_{d-1} + q_d, p_d] \right\} \\ &= \sum_{i \geq d-r} \left(\frac{\partial u_{n-d}}{\partial p_i} \frac{\partial u_{n-r-1}}{\partial q_i} - \frac{\partial u_{n-d}}{\partial q_i} \frac{\partial u_{n-r-1}}{\partial p_i} \right). \end{aligned}$$

Then, we anticipate results in §3 by setting $v_n := u_{n-d} - u_n$, to find that

$$\begin{aligned} \frac{\partial u_{n-d}}{\partial p_i} &= (-1)^i v_n^2 \dots v_{n+i-1}^2, \quad i < d, \\ \frac{\partial u_{n-d}}{\partial p_d} &= (-1)^d v_n^2 \dots v_{n+d-1}^2 - v_n^2 v_{n+d-1}^2, \\ \frac{\partial u_{n-d}}{\partial q_i} &= (-1)^{i-1} v_n^2 \dots v_{n+i-2}^2 (1 - v_{n+i-1}^2), \quad i < d, \\ \frac{\partial u_{n-d}}{\partial q_d} &= v_n^2 + (-1)^{d-1} v_n^2 \dots v_{n+d-2}^2, \end{aligned}$$

$$\begin{aligned} \frac{\partial u_{n-r-1}}{\partial p_i} &= (-1)^{r+i-1-d} v_{n+d-r-1}^2 \dots v_{n+i-1}^2, \\ \frac{\partial u_{n-r-1}}{\partial q_i} &= (-1)^{r+i-d} v_{n+d-r-1}^2 \dots v_{n+i-2}^2 (1 - v_{n+i-1}^2), \quad i < d \\ \text{and } \frac{\partial u_{n-r-1}}{\partial q_d} &= (-1)^r v_{n+d-r-1}^2 \dots v_{n+d-2}^2, \end{aligned}$$

for $i \geq d - r \geq 2$, and hence we obtain $\{u_{n-d}, u_{n-r-1}\} = 0$. Next, we use $\{u_{n-d}, u_{n+j}\} = \{u_{n-d}, q_{j+1}\} = \partial u_{n-d} / \partial p_{j+1}$, where $0 \leq j \leq d - 1$, from which the second and third formulae in (2.9) follow. ■

3. Poisson structures and tau-functions for the Korteweg–de Vries maps

As mentioned earlier, the discrete KdV equation can be derived from the double pKdV equation. This suggests that the symplectic structure given in §2 can be used to find a Poisson structure for the $(d - 1, -1)$ -reduction of the discrete KdV equation. It turns out that, in addition to the bracket induced from double pKdV, each of the KdV maps has a second, independent Poisson bracket, which is obtained from a Hirota bilinear form in terms of tau-functions. The second bracket is constructed by making use of a connection with Somos recurrences and cluster algebras.

(a) First Poisson structure from potential Korteweg–de Vries

Whenever u_n is a solution of equation (2.7), $v_n = u_{n-d} - u_n$ satisfies a difference equation of order d , namely

$$v_{n+d-1} - v_{n+1} - \frac{1}{v_{n+d}} + \frac{1}{v_n} = 0. \quad (3.1)$$

Alternatively, by starting from a solution of (1.3) with the periodicity property (1.5), we see that this yields a solution $v = v_{\ell,m}$ of (1.4) with the same periodicity, and so (writing this as v_n , with the same abuse of notation as before) it is clear that equation (3.1) is just the $(d-1, -1)$ -reduction of the discrete KdV equation. Equivalently, the ordinary difference equation (3.1) corresponds to the d -dimensional map

$$\varphi : (v_0, v_1, \dots, v_{d-1}) \mapsto \left(v_1, v_2, \dots, v_{d-1}, \frac{v_0}{1 + v_{d-1}v_0 - v_1v_0} \right). \quad (3.2)$$

The case $d=2$ is trivial, so henceforth we consider $d \geq 3$.

In the above, the suffix n has been dropped, taking a fixed d -tuple $(v_0, v_1, \dots, v_{d-1})$ in d -dimensional space. However, because the map (3.2) is obtained from a recurrence of order d , all of the formulae are invariant under simultaneous shifts of all indices by an arbitrary amount n , i.e. $v_j \rightarrow v_{n+j}$ for each j . For a fixed n , say $n=0$, the formulae in theorem 2.1 define a Poisson bracket in dimension $2d$, which can be used to calculate the brackets between the quantities $v_j = u_{j-d} - u_j$ for $j=0, \dots, d-1$. Remarkably, these brackets can be rewritten in terms of v_j alone; in other words, these quantities form a Poisson subalgebra of dimension d . Hence, this provides the first of two ways to endow (3.2) with a Poisson structure.

Theorem 3.1. *The d -dimensional map (3.2) preserves the Poisson bracket $\{, \}_1$ defined by*

$$\{v_i, v_j\}_1 = \begin{cases} (-1)^{j-i} \prod_{r=i}^j v_r^2, & 0 < j-i < d-1, \\ \left(1 + (-1)^{d-1} \prod_{r=1}^{d-2} v_r^2 \right) v_0^2 v_{d-1}^2, & j-i = d-1. \end{cases} \quad (3.3)$$

This bracket is degenerate, with one Casimir when d is odd, and two independent Casimirs when d is even.

The above result follows immediately from theorem 2.1, apart from the statement about the Casimirs, which will be explained shortly. Here, we first give a couple of examples for illustration.

Example 3.2. When $d=3$, the map φ given by (3.2) preserves the bracket

$$\{v_0, v_1\}_1 = -v_0^2 v_1^2, \quad \{v_1, v_2\}_1 = -v_1^2 v_2^2, \quad \{v_0, v_2\}_1 = (1 + v_1^2) v_0^2 v_2^2.$$

This Poisson bracket has rank two, with a Casimir \mathcal{C} that is also a first integral for the map, i.e. $\varphi^* \mathcal{C} = \mathcal{C}$ with

$$\mathcal{C} = v_1 - \frac{1}{v_0} - \frac{1}{v_1} - \frac{1}{v_2}.$$

Example 3.3. When $d=4$, the map (3.2) preserves the bracket specified by

$$\{v_0, v_1\}_1 = -v_0^2 v_1^2, \quad \{v_0, v_2\}_1 = v_0^2 v_1^2 v_2^2, \quad \{v_0, v_3\}_1 = (1 - v_1^2 v_2^2) v_0^2 v_3^2,$$

where all other brackets $\{v_i, v_j\}_1$ for $0 \leq i, j \leq d-1$ are determined from skew-symmetry and the Poisson property of φ . This is a bracket of rank two, having two independent Casimirs given by

$$\mathcal{C}_1 = v_1 - \frac{1}{v_0} - \frac{1}{v_2}, \quad \mathcal{C}_2 = v_2 - \frac{1}{v_1} - \frac{1}{v_3}, \quad \text{with } \varphi^* \mathcal{C}_1 = \mathcal{C}_2, \quad \varphi^* \mathcal{C}_2 = \mathcal{C}_1.$$

Remark 3.4. The Casimirs \mathcal{C}_j in the preceding example are 2-integrals [25], meaning that they are preserved by two iterations of the map, i.e. $(\varphi^*)^2 \mathcal{C}_j = \mathcal{C}_j$ for $j=1, 2$. The symmetric functions

$$\mathcal{K} = \mathcal{C}_1 + \mathcal{C}_2, \quad \mathcal{K}' = \mathcal{C}_1 \mathcal{C}_2 \quad (3.4)$$

provide two independent first integrals.

In order to make the properties of the Poisson bracket $\{, \}_1$ more transparent, we introduce some new coordinates, the motivation for which should become clear from the Lax pairs in §4.

Lemma 3.5. For all $d \geq 4$, with respect to the coordinates

$$g_0 = -\frac{1}{v_0}, \quad g_j = v_{j-1} - \frac{1}{v_j}, \quad j = 1, \dots, d-1, \quad (3.5)$$

the first Poisson bracket for the KdV map (3.2) is specified by the following relations for $v = 1$ and $0 \leq i < j \leq d-1$:

$$\{g_i, g_j\}_1 = \begin{cases} -1, & \text{if } j - i = 1, \\ 1, & \text{if } j - i = d - 1, \\ \frac{v}{g_0^2}, & \text{if } i = 1, j = d - 1, \\ 0, & \text{otherwise.} \end{cases} \quad (3.6)$$

When d is odd, this bracket has the Casimir

$$\mathcal{C} = g_0 + g_1 + \dots + g_{d-1} + \frac{v}{g_0}, \quad (3.7)$$

whereas for even d , there are the two Casimirs

$$\mathcal{C}_1 = g_0 + g_2 + \dots + g_{d-2}, \quad \mathcal{C}_2 = g_1 + g_3 + \dots + g_{d-1} + \frac{v}{g_0}. \quad (3.8)$$

Remark 3.6. When $v = 0$, the Poisson bracket (3.6) is just the first Poisson bracket for the dressing chain, as given by equation (13) in Veselov & Shabat [27].

In terms of the coordinates g_j , the map (3.2) is rewritten as

$$\varphi: (g_0, g_1, \dots, g_{d-1}) \mapsto \left(g_1 + \frac{1}{g_0}, g_2, \dots, g_{d-1}, \frac{g_0^2 g_1}{1 + g_0 g_1} \right). \quad (3.9)$$

Note that for the special case $d = 3$, as in example 3.2, part of the formula for the bracket (3.6) requires a slight modification, namely $\{g_1, g_2\}_1 = -1 + 1/g_0^2$.

(b) Second Poisson structure from cluster algebras for tau-functions

The discrete KdV equation (1.4) was derived by Hirota in terms of tau-functions, via the Bäcklund transformation for the differential–difference KdV equation [28]. In Hirota's approach, the solution of the discrete KdV equation is given in terms of a tau-function as $v = \tilde{\tau} \hat{\tau} / (\tau \hat{\tilde{\tau}})$, and at the level of the $(d-1, -1)$ -reduction, this becomes

$$v_n = \frac{\tau_{n+1} \tau_{n+d-1}}{\tau_n \tau_{n+d}}. \quad (3.10)$$

By direct substitution, it then follows that v_n is a solution of (3.1) provided that τ_n satisfies the trilinear (degree three) recurrence relation

$$\tau_{n+2d} \tau_{n+d-1} \tau_{n+1} = \tau_{n+2d-1} \tau_{n+d+1} \tau_n - \tau_{n+2d-1} \tau_{n+d-1} \tau_{n+2} + \tau_{n+2d-2} \tau_{n+d+1} \tau_{n+1}. \quad (3.11)$$

However, this relation can be further simplified upon dividing by $\tau_{n+1} \tau_{n+d}$, which gives the relation

$$\alpha_n = \frac{\tau_n \tau_{n+d+1} - \tau_{n+2} \tau_{n+d-1}}{\tau_{n+1} \tau_{n+d}} = \frac{\tau_{n+d-1} \tau_{n+2d} - \tau_{n+d+1} \tau_{n+2d-2}}{\tau_{n+d} \tau_{n+2d-1}} = \alpha_{n+d-1}.$$

This immediately yields relations that are bilinear (degree two) in τ_n .

Proposition 3.7. The solutions of the equation (3.1) are given in terms of a tau-function by (3.10), where τ_n satisfies the bilinear recurrence relation

$$\tau_{n+d+1} \tau_n = \alpha_n \tau_{n+d} \tau_{n+1} + \tau_{n+d-1} \tau_{n+2} \quad (3.12)$$

of order $d+1$, with the coefficient α_n having period $d-1$.

Apart from the presence of the periodic coefficient α_n , the bilinear relation (3.12) has the form of a Somos- $(d + 1)$ recurrence [29]. Such recurrence relations (with constant coefficients) are also referred to as three-term Gale–Robinson recurrences (after [30] and [31], respectively).

Example 3.8. For $d = 3$, the equation (3.12) is a Somos-4 recurrence with coefficients of period 2, that is $\tau_{n+4}\tau_n = \alpha_n\tau_{n+3}\tau_{n+1} + \tau_{n+2}^2$, with $\alpha_{n+2} = \alpha_n$. Due to the Laurent phenomenon [32], the iterates of this recurrence are Laurent polynomials, i.e. polynomials in the initial values data and their reciprocals with integer coefficients; to be precise, $\tau_n \in \mathbb{Z}[\alpha_0, \alpha_1, \tau_0^{\pm 1}, \tau_1^{\pm 1}, \tau_2^{\pm 1}, \tau_3^{\pm 1}]$ for all $n \in \mathbb{Z}$. This means that integer sequences can be generated from a suitable choice of initial data and coefficients. For instance, with the initial values $\tau_0 = \tau_1 = \tau_2 = \tau_3 = 1$ and parameters $\alpha_0 = 1, \alpha_1 = 2$, the Somos-4 recurrence yields an integer sequence beginning with 1, 1, 1, 1, 2, 5, 9, 61, 193, 1439, 13892, 121853, 1908467, 47166783, ...

From the work of Fordy & Marsh [33], it is known that, at least in the case where the coefficients are constant, recurrences of Somos type can be generated from sequences of mutations in a cluster algebra. For the purposes of this paper, the main advantage of considering the cluster algebra is that it provides a natural presymplectic structure for the tau-functions. A presymplectic form that is compatible with cluster mutations was presented in Gekhtman *et al.* [20], and in Fordy & Hone [22], it was explained how this presymplectic structure is preserved by the recurrences considered in Fordy & Marsh [33].

Cluster algebras are a new class of commutative algebras that were introduced in Fomin & Zelevinsky [34]. Rather than having a set of generators and relations that are given from the start, the generators of a cluster algebra are defined recursively by an iterative process known as cluster mutation. For a coefficient-free cluster algebra, one starts from an initial set of generators (the initial cluster) of fixed size, which here we take to be $d + 1$. If the initial cluster is denoted by $(\tau_1, \dots, \tau_{d+1})$, then for each index k , one defines the mutation in the k -direction to be the transformation that exchanges one of the variables to produce a new cluster $(\tau'_1, \dots, \tau'_{d+1})$ given by

$$\tau'_j = \tau_j, \quad j \neq k, \quad \tau'_k \tau_k = \prod_{j=1}^{d+1} \tau_j^{[b_{jk}]_+} + \prod_{j=1}^{d+1} \tau_j^{[-b_{jk}]_+}, \quad (3.13)$$

where the exponents in the exchange relation for τ'_k come from an integer matrix $B = (b_{ij})$ known as the *exchange matrix*, and we have used the notation $[b]_+ = \max(b, 0)$.

As well as cluster mutation, there is an associated operation of matrix mutation, which acts on the matrix B ; the details of this are omitted here. Fordy and Marsh gave conditions under which skew-symmetric exchange matrices B have a cyclic symmetry (or periodicity) under mutation, and classified all such B with period 1 [33]. They also showed how this led to recurrence relations for cluster variables, by taking cyclic sequences of mutations. The requirement of periodicity puts conditions on the elements of the skew-symmetric matrix B , which (for a suitable labelling of indices) can be written as

$$b_{i,d+1} = b_{1,i+1}, \quad i = 1, \dots, d, \quad (3.14)$$

and

$$b_{i+1,j+1} = b_{ij} + b_{1,i+1}[-b_{1,j+1}]_+ - b_{1,j+1}[-b_{1,i+1}]_+, \quad 1 \leq i, j \leq d. \quad (3.15)$$

The corresponding recurrence is defined by iteration of the map $(\tau_1, \dots, \tau_d, \tau_{d+1}) \mapsto (\tau_2, \dots, \tau_{d+1}, \tau'_1)$ associated with the exchange relation (3.13) for index $k = 1$, where the exponents are given by the entries $b_{1,j}$ in the first row of B . Moreover, given a recurrence relation of this type, the conditions (3.14) and (3.15) allow the rest of the matrix B to be constructed from the exponents corresponding to the first row, and these conditions are also necessary and sufficient for a log-canonical presymplectic form ω , as in (3.16) below, to be preserved (see lemma 2.3 in Fordy & Hone [22]). In general, this two-form is closed, but it may be degenerate.

For the case at hand, the exponents appearing in the two monomials on the right-hand side of (3.12) specify the first row of the matrix B as $(0, 1, -1, 0, \dots, 0, -1, 1)$, and the rest of this matrix is found by applying (3.14) and (3.15). Although the foregoing discussion was put in the context of

coefficient-free cluster algebras, the presence of coefficients in front of these two monomials does not affect the behaviour of the corresponding log-canonical two-form under iteration. Thus, we obtain

Lemma 3.9. *For all $d \geq 5$, the Somos- $(d + 1)$ recurrence (3.12) preserves the presymplectic form*

$$\omega = \sum_{i < j} \frac{b_{ij}}{\tau_i \tau_j} d\tau_i \wedge d\tau_j, \quad (3.16)$$

given in terms of the entries of the associated skew-symmetric exchange matrix $B = (b_{ij})$ of size $d + 1$, where (with an asterisk denoting the omitted entries below the diagonal)

$$B = \begin{pmatrix} 0 & 1 & -1 & 0 & \cdots & 0 & -1 & 1 \\ & \ddots & 2 & -1 & 0 & \cdots & 1 & -1 \\ & & \ddots & \ddots & \ddots & \ddots & & 0 \\ & & & \ddots & \ddots & \ddots & \ddots & \vdots \\ & & & & \ddots & \ddots & \ddots & 0 \\ & & & & & \ddots & 2 & -1 \\ * & & & & & & 0 & 1 \\ & & & & & & & 0 \end{pmatrix}.$$

Remark 3.10. In each of the special cases $d = 3, 4$, the exchange matrix does not fit into the general pattern above. For the details of the case $d = 3$, when (3.12) is a Somos-4 relation, see example 2.11 in Fordy & Hone [22]; and for $d = 4$ (Somos-5), see example 1.1 in Fordy & Hone [21].

The exchange matrices being considered here are all degenerate: for d odd, B has a two-dimensional kernel, whereas for d even, the kernel is three-dimensional. In the odd case, the kernel is associated with the action of a two-parameter group of scaling symmetries, namely

$$\tau_n \rightarrow \rho \sigma^n \tau_n, \quad \rho, \sigma \neq 0, \quad (3.17)$$

whereas in the even case, there is the additional symmetry

$$\tau_n \rightarrow \xi^{(-1)^n} \tau_n, \quad \xi \neq 0. \quad (3.18)$$

In the theory of tau-functions, such scalings are known as gauge transformations. By lemma 2.7 in Fordy & Hone [22] (see also §6 therein), symplectic coordinates are obtained by taking a complete set of invariants for these scaling symmetries. With a suitable choice of coordinates, denoted below by y_n , the associated symplectic maps can be written in the form of recurrences, like so:

- **Odd d :** the quantities $y_n = \tau_{n+2}\tau_n/\tau_{n+1}^2$ are invariant under (3.17), and satisfy the difference equation

$$y_{n+d-1}(y_{n+d-2} \cdots y_{n+1})^2 y_n = \alpha_n y_{n+d-2} \cdots y_{n+1} + 1. \quad (3.19)$$

- **Even d :** the quantities $y_n = \tau_{n+3}\tau_n/(\tau_{n+2}\tau_{n+1})$ are invariant under both (3.17) and (3.18), and satisfy the difference equation

$$y_{n+d-2} \cdots y_n = \alpha_n y_{n+d-3} y_{n+d-5} \cdots y_{n+1} + 1. \quad (3.20)$$

Example 3.11. Both of the cases, $d = 3$ and $d = 4$, lead to iteration of maps in the plane with coefficients that vary periodically, namely

$$d = 3: y_{n+2}y_{n+1}^2y_n = \alpha_n y_{n+1} + 1, \quad \text{with } \alpha_{n+2} = \alpha_n,$$

and

$$d = 4: y_{n+2}y_{n+1}y_n = \alpha_n y_{n+1} + 1, \quad \text{with } \alpha_{n+3} = \alpha_n.$$

Remark 3.12. Owing to the presence of the periodic coefficients, the latter maps are not of standard Quispel–Roberts–Thompson (QRT) type [3], but they reduce to symmetric QRT maps when $\alpha_n = \text{constant}$. Maps in the plane of this more general type have recently been studied systematically by Roberts [35].

In general, the solutions of (3.19) or (3.20) correspond to the iterates of a symplectic map, in dimension $d - 1$ or $d - 2$, respectively. To be more precise, rather than just iterating a single map, each iteration depends on the coefficient α_n that varies with a fixed period (as in proposition 3.7), but the same symplectic structure is preserved at each step. The appropriate symplectic form in the coordinates y_n can be obtained from (3.16), and by a direct calculation the associated non-degenerate bracket is found; this is presented as follows.

Lemma 3.13. *For d odd, each iteration of (3.19) preserves the non-degenerate Poisson bracket specified by*

$$\{y_i, y_j\} = (-1)^{j-i+1} y_i y_j, \quad 0 \leq i < j \leq d - 2. \quad (3.21)$$

For d even, the map defined by (3.20) preserves the non-degenerate bracket in dimension $d - 2$ given by

$$\{y_i, y_{i\pm 1}\} = \pm y_i y_{i\pm 1}, \quad (3.22)$$

with all other brackets $\{y_i, y_j\}$ for $0 \leq i < j \leq d - 3$ being zero.

The bracket for the variables y_i is the key to deriving a second Poisson structure for the KdV maps when d is odd.

Theorem 3.14. *In the case that d is odd, the map (3.2) preserves a second Poisson bracket, which is specified in terms of the coordinates (3.5) by*

$$\{g_i, g_j\}_2 = (-1)^{j-i+1} g_i g_j, \quad 0 \leq i < j \leq d - 1. \quad (3.23)$$

Proof. Substituting for v_n from the formula (3.10) and making use of the bilinear equation (3.12) produces the identities

$$v_{j-1} - \frac{1}{v_j} = \frac{\tau_j(\tau_{j+1}\tau_{j+d-2} - \tau_{j-1}\tau_{j+d})}{\tau_{j+d-1}\tau_{j-1}\tau_{j+1}} = -\frac{\alpha_{j-1}\tau_j^2}{\tau_{j-1}\tau_{j+1}},$$

so that, by the definition of the symplectic coordinates y_j , $g_j = -\alpha_{j-1}/y_{j-1}$ for $j = 1, \dots, d - 1$. A similar calculation in terms of tau-functions also yields $g_0 = -y_0 y_1 \cdots y_{d-2}$. Noting that the coefficients α_i play the role of constants with respect to the bracket (3.21), this immediately implies that the induced brackets between the g_i are

$$\{g_i, g_j\} = \alpha_{i-1}\alpha_{j-1}y_{i-1}^{-2}y_{j-1}^{-2}\{y_{i-1}, y_{j-1}\} = (-1)^{j-i+1} g_i g_j$$

for $1 \leq i < j \leq d - 1$, which agrees with (3.23). By making use of the preceding formula for g_0 in terms of y_i , the brackets $\{g_0, g_j\}$ follow in the same way. ■

Remark 3.15. The bracket (3.23) is the same as the quadratic bracket for the dressing chain (for the case where all parameters β_i are zero in Veselov & Shabat [27]). It has the Casimir

$$C^* = g_0 g_1 \cdots g_{d-1}, \quad (3.24)$$

which is also given in terms of the coefficients of the bilinear equation (3.12) by $C^* = -\prod_{j=0}^{d-2} \alpha_j$.

The case where d is even is slightly more complicated, because we do not have a direct way to derive a second Poisson bracket in terms of the coordinates g_i (or equivalently v_i). The reason for this difficulty is that, from the tau-function expressions, although the quantities g_i remain the same under the action of the two-parameter symmetry group (3.17), they are not invariant under the additional scaling (3.18), but rather they transform differently according to the parity of the index i :

$$g_i \longrightarrow \xi^{\pm 4} g_i \quad \text{for even/odd } i. \quad (3.25)$$

In order to obtain a fully invariant set of variables, we introduce a projection π from dimension d to dimension $d - 1$:

$$\pi : f_i = g_i g_{i+1}, \quad i = 0, \dots, d - 2.$$

By regarding the new coordinates f_i as functions of the g_j , we get an induced map φ' in dimension $d - 1$, which is compatible with φ in the sense that $\varphi \cdot \pi = \pi \cdot \varphi'$; this has the form

$$\varphi' : (f_0, f_1, \dots, f_{d-2}) \mapsto \left(f_1(1 + f_0^{-1}), f_2, \dots, f_{d-2}, \frac{f_0 f_2 \cdots f_{d-2}}{(1 + f_0^{-1}) f_1 f_3 \cdots f_{d-3}} \right). \quad (3.26)$$

In terms of tau-functions, the quantities f_i are invariant under the action of the full three-parameter group of gauge transformations, which means that they can be expressed as functions of the invariant symplectic coordinates y_n , and hence we can derive a Poisson bracket for them.

Theorem 3.16. *In the case that d is even, the map (3.26) preserves a Poisson bracket, which is specified in terms of the coordinates f_i by*

$$\{f_i, f_{i\pm 1}\}_2 = \pm f_i f_{i\pm 1}, \quad (3.27)$$

with all other brackets $\{f_i, f_j\}_2$ for $0 \leq i < j \leq d - 2$ being zero.

Proof. Following the proof of theorem 3.14, we have $g_i g_{i+1} = \alpha_{i-1} \alpha_i \tau_i \tau_{i+1} / (\tau_{i-1} \tau_{i+2})$, whence, in terms of the symplectic coordinates for (3.20), we have $f_i = \alpha_{i-1} \alpha_i / y_{i-1}$ for $i = 1, \dots, d - 2$. A similar calculation yields a slightly different formula for index $i = 0$: $f_0 = \alpha_0 y_1 y_3 \cdots y_{d-3}$, and then the Poisson brackets (3.22) between the y_j directly imply that the brackets (3.27) hold between the f_i . ■

Remark 3.17. The bracket (3.27) has the Casimir

$$C^* = f_0 f_2 \cdots f_{d-2}, \quad (3.28)$$

which is also given in terms of the coefficients of the bilinear equation (3.12) by $C^* = \prod_{j=0}^{d-2} \alpha_j$.

It may be unclear why there is the suffix 2 on the bracket in (3.27), because we have not yet provided another Poisson bracket for the map (3.26). However, as will be explained in §4, when d is even the quantities f_i form a Poisson subalgebra for the bracket $\{, \}_1$ of theorem 3.1. As will also be explained, the brackets $\{, \}_1$ and $\{, \}_2$ are compatible (in the sense that any linear combination of them is also a Poisson bracket), which means that a standard bi-Hamiltonian argument can be used to show Liouville integrability of the maps for either odd or even d .

4. First integrals and integrability of the Korteweg–de Vries maps

The purpose of this section is to prove the following result.

Theorem 4.1. *For each $d \geq 3$, the map (3.2) is completely integrable in the Liouville sense.*

Recall that, for a Poisson map φ in dimension d , Liouville integrability means that there should be k Casimirs invariant under the map, so that the $(d - k)$ -dimensional symplectic leaves of the Poisson bracket are preserved by φ , plus an additional $\frac{1}{2}(d - k)$ independent first integrals that are in involution with respect to the bracket. For the particular map in question, there is always the Poisson bracket $\{, \}_1$, and from lemma 3.5, this has either one or two Casimirs, with symplectic leaves of dimension $d - 1$ or $d - 2$, for odd/even d , respectively; hence, an additional $\lfloor \frac{1}{2}(d - 1) \rfloor$ first integrals are required in this case.

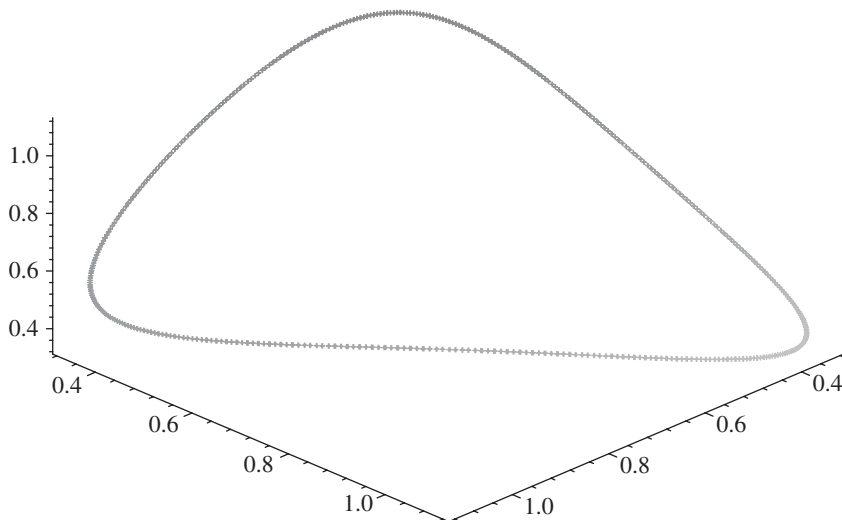


Figure 1. The orbit of the map (3.2) for $d = 3$ with initial data $(v_0, v_1, v_2) = (1, \frac{1}{2}, \frac{2}{5})$.

For the lowest values $d = 3, 4$, it is straightforward to verify complete integrability, because in those cases, only one extra first integral is required, apart from the Casimirs of $\{, \}_1$. When $d = 3$, as in example 3.2, the Casimir \mathcal{C} is preserved by φ ; the quantity \mathcal{C}^* in (3.24), which is the Casimir of the second bracket, is also a first integral. Similarly, for $d = 4$, the two Casimirs in example 3.3 provide the two first integrals (3.4), which are themselves Casimirs; the quantity $\mathcal{C}^* = f_0 f_2 = g_0 g_1 g_2 g_3$, as in (3.28), provides the extra first integral.

Example 4.2. For $d = 3$, each of the first integrals $\mathcal{C}, \mathcal{C}^*$ of the map (3.2) define surfaces in three dimensions, given by

$$XY^2Z - CXYZ - XY - YZ - ZX = 0$$

and

$$XY^2Z + \mathcal{C}^*XYZ - XY - YZ + 1 = 0,$$

respectively, in terms of coordinates $(X, Y, Z) \equiv (v_n, v_{n+1}, v_{n+2})$. These two surfaces intersect in a curve of genus one, found explicitly by eliminating the variable Z above; this yields the biquadratic

$$(\mathcal{C} + \mathcal{C}^*)X^2Y^2 + (X + Y)(XY - 1) - CXY = 0. \quad (4.1)$$

The embedding of such a curve in three dimensions can be seen from the orbit plotted in figure 1.

For higher values of d , it is not so obvious how to proceed, but the correct number of additional first integrals can be obtained by constructing a Lax pair for the map, as we now describe.

(a) Lax pairs and monodromy

The discrete KdV equation (1.4) is known to be integrable in the sense that it arises as the compatibility condition for a pair of linear equations. In Nijhoff [36], a scalar Lax pair is given as follows:

$$\hat{\phi} = v\tilde{\phi} + \lambda\phi, \quad \hat{\phi} = \tilde{\phi} + \frac{1}{v}\phi.$$

Upon introducing the vector $\Phi := \begin{pmatrix} \tilde{\phi} \\ \phi \end{pmatrix}$, the latter pair of scalar equations leads to a Lax pair in matrix form, namely

$$\tilde{\Phi} = \mathbf{L}\Phi, \quad \hat{\Phi} = \mathbf{M}\Phi, \quad \text{with } \mathbf{L} = \begin{pmatrix} v - \frac{1}{v} & \lambda \\ 1 & 0 \end{pmatrix}, \quad \mathbf{M} = \begin{pmatrix} v & \lambda \\ 1 & \frac{1}{v} \end{pmatrix}. \quad (4.2)$$

The equation (1.4) is equivalent to the compatibility condition for the linear system (4.2), that is

$$\hat{\mathbf{L}}\mathbf{M} = \tilde{\mathbf{M}}\mathbf{L}. \quad (4.3)$$

Note that the matrix \mathbf{L} is associated with shifts in the ℓ (horizontal) direction, and \mathbf{M} with shifts in the m (vertical) direction.

First integrals of each KdV map (3.2) can be found by using the staircase method [4,13–15]. For the $(d-1, -1)$ -reduction, a staircase on the \mathbb{Z}^2 lattice is built from paths consisting of $d-1$ horizontal steps and one vertical step. Taking an ordered product of Lax matrices along the staircase yields the monodromy matrix

$$\mathcal{L}_n = \mathbf{M}_n^{-1} \mathbf{L}_{n+d-2} \cdots \mathbf{L}_n, \quad (4.4)$$

corresponding to $d-1$ steps to the right ($\ell \rightarrow \ell+1$) and one step down ($m \rightarrow m-1$), where $\mathbf{L} \rightarrow \mathbf{L}_n$ and $\mathbf{M} \rightarrow \mathbf{M}_n$ under the reduction. As a consequence of (4.3), the identity $\mathbf{L}_{n+d-1} \mathbf{M}_n = \mathbf{M}_{n+1} \mathbf{L}_n$ holds, which implies that \mathcal{L}_n satisfies the discrete Lax equation

$$\mathcal{L}_{n+1} \mathbf{L}_n = \mathbf{L}_n \mathcal{L}_n. \quad (4.5)$$

The latter holds if and only if v_n satisfies the difference equation (3.1). Because (4.5) means that the spectrum of \mathcal{L}_n is invariant under the shift $n \rightarrow n+1$, first integrals for the KdV map can be constructed from the trace of the monodromy matrix (or powers thereof), which can be expanded in the spectral parameter λ .

For convenience, we conjugate \mathcal{L}_n in (4.4) by \mathbf{M}_n and multiply by an overall factor of $\lambda-1$, which (upon setting $n=0$) gives a modified monodromy matrix

$$\mathcal{L}^\dagger = \mathbf{L}_{d-1}^\dagger \cdots \mathbf{L}_1^\dagger \mathbf{L}_0^\dagger, \quad (4.6)$$

where

$$\mathbf{L}_i^\dagger = \mathbf{L}_{i-1} = \begin{pmatrix} g_i & \lambda \\ 1 & 0 \end{pmatrix}, \quad i=1, \dots, d-1, \quad \text{but } \mathbf{L}_0^\dagger = \begin{pmatrix} g_0 & \lambda \\ 1 & \frac{v}{g_0} \end{pmatrix} \quad \text{with } v=1.$$

Thus, we see the origin of the coordinates g_i introduced in lemma 3.5: they are the $(1, 1)$ entries in the Lax matrices that make up the monodromy matrix. We have already mentioned a connection with the dressing chain at the level of the Poisson brackets, but it can be seen more directly here: up to taking an inverse and inserting a spectral parameter, when $v=0$ the expression (4.6) reduces to the monodromy matrix for the dressing chain found in Fordy & Hone [22]. The parameter v can be regarded as introducing inhomogeneity into the chain at $i=0$ (and for $v \neq 0$ it is always possible to rescale so that $v=1$).

With the introduction of another spectral parameter μ , we consider the characteristic polynomial of \mathcal{L}^\dagger , which defines the spectral curve

$$\chi(\lambda, \mu) := \det(\mathcal{L}^\dagger - \mu \mathbf{1}) = \mu^2 - \mathcal{P}(\lambda)\mu + (v-\lambda)(-\lambda)^{d-1} = 0, \quad (4.7)$$

with $\mathcal{P}(\lambda) = \text{tr} \mathcal{L}^\dagger$. This curve in the (λ, μ) plane is invariant under the KdV map (3.2), and the trace of \mathcal{L}^\dagger provides the polynomial \mathcal{P} whose non-trivial coefficients are first integrals of the map.

Example 4.3. When $d=3$, the spectral curve is of genus one, given by $\mu^2 - (C^* + C\lambda)\mu + (v-\lambda)\lambda^2 = 0$. For $v=1$, this is isomorphic to the biquadratic curve (4.1) corresponding to the intersection of the level sets of C and C^* .

Example 4.4. When $d=4$, the spectral curve also has genus one, being given by $\mu^2 - (I_0 + I_1\lambda + 2\lambda^2)\mu - (v-\lambda)\lambda^3 = 0$, with coefficients expressed in terms of g_i by

$$I_0 = C^* = g_0 g_1 g_2 g_3, \quad I_1 = \mathcal{K}' - v = g_0 g_1 + g_1 g_2 + g_2 g_3 + g_3 g_0 + v \frac{g_2}{g_0}.$$

In §4*b*, we will show how, for each d , expanding the trace of the monodromy matrix in powers of λ gives the polynomial \mathcal{P} of degree $\lfloor d/2 \rfloor$. This implies that the hyperelliptic curve $\chi(\lambda, \mu) = 0$ as in (4.7) has genus $\lfloor \frac{1}{2}(d-1) \rfloor$. Thus, we expect that the (real, compact) Liouville tori for the KdV

map should be identified with (a real component of) the Jacobian variety of this curve, because their dimensions coincide, and the map should correspond to a translation on the Jacobian.

(b) Direct proof from the Vieta expansion

In order to calculate the trace of the monodromy matrix explicitly, we split the Lax matrices as

$$\mathbf{L}_i^\dagger = \lambda \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} g_i & 0 \\ 1 & 0 \end{pmatrix} = \lambda \mathbf{X} + \mathbf{Y}_i, \quad i = 1, \dots, d-1, \quad (4.8)$$

and similarly for \mathbf{L}_0^\dagger . This splitting fits into the framework of Tran *et al.* [16], with the coefficient of λ being the same nilpotent matrix \mathbf{X} in each case, and allows the application of the so-called Vieta expansion.

Recall that, in the non-commutative setting, the Vieta expansion is given by the formula

$$\prod_{i=a}^{\widehat{b}} (\lambda \mathbf{X}_i + \mathbf{Y}_i) := (\lambda \mathbf{X}_b + \mathbf{Y}_b) \dots (\lambda \mathbf{X}_a + \mathbf{Y}_a) = \sum_{r=0}^{b-a+1} \lambda^r \mathbf{Z}_r^{a,b}, \quad (4.9)$$

where

$$\mathbf{Z}_r^{a,b} = \sum_{a \leq i_1 < i_2 < \dots < i_r \leq b} \mathbf{Y}_b \mathbf{Y}_{b-1} \dots \mathbf{Y}_{i_r+1} \mathbf{X}_{i_r} \mathbf{Y}_{i_r-1} \dots \mathbf{Y}_{i_1+1} \mathbf{X}_{i_1} \mathbf{Y}_{i_1-1} \dots \mathbf{Y}_a.$$

In the case at hand, we have $\mathbf{X}_i = \mathbf{X}$ for all i , and by writing the monodromy matrix in the form (4.9), we can use lemma 8 in Tran *et al.* [16] to expand the trace as

$$\mathcal{P}(\lambda) = \text{tr} \mathcal{L}^\dagger = \sum_{r=0}^{\lfloor d/2 \rfloor} I_r \lambda^r, \quad (4.10)$$

where the first integrals of the KdV map are given by the closed-form expression

$$I_r = \Psi_{r-1}^{1,d-3} + g_0 \Psi_{r-1}^{1,d-4} + \left(g_{d-1} + \frac{\nu}{g_0} \right) \Psi_{r-1}^{2,d-3} + \Psi_{r-2}^{2,d-4} + g_0 g_{d-1} \Psi_r^{1,d-3}, \quad (4.11)$$

with

$$\Psi_r^{a,b} = \prod_{i=a}^{b+1} g_i \left(\sum_{a \leq i_1 < i_1+1 < i_2 < \dots < i_r \leq b} \prod_{j=1}^r \frac{1}{g_{i_j} g_{i_j+1}} \right). \quad (4.12)$$

The explicit form of these integrals mean that it is possible to give a direct verification that they are in involution, which is almost identical to the proof of theorem 13 in Tran *et al.* [19].

Proposition 4.5. *The first integrals (4.11) of the KdV map Poisson commute with respect to the bracket $\{, \}_1$ in (3.3).*

Proof. Writing the Poisson structure in terms of the coordinates g_i as in lemma 3.5, we see that (up to an overall factor of -1) the brackets $\{g_i, g_j\}_1$ in (3.6) for $1 \leq i, j \leq d-2$ are identical to the brackets between the coordinates c_i given by equation (46) in Tran *et al.* [19]. The polynomial functions given by (4.12) are the same as in the latter reference, and the particular functions $\Psi_r^{a,b}$ that appear in the formula (4.11) only depend on g_i for $1 \leq i \leq d-2$. This implies that the brackets between these functions are the same as those given in lemma 11 and corollary 12 in Tran *et al.* [19], whence it is straightforward to verify that $\{I_r, I_s\}_1 = 0$ for $0 \leq r, s \leq \lfloor d/2 \rfloor$. ■

In order to complete the proof of theorem 4.1, it only remains to check that there are sufficiently many independent integrals.

For odd d , the leading ($\nu = 0$) part of the first integral I_r is a cyclically symmetric, homogeneous function of degree $d - 2r$ in the g_i , being identical to the independent integrals of the dressing chain in Veselov *et al.* [27] (for parameters $\beta_i = 0$), with terms of each odd degree appearing for $r = 0, \dots, (d-1)/2$. Hence, these functions are also independent in the case $\nu \neq 0$, corresponding

to the addition of a rational term with g_0 in the denominator, which is linear in v and of degree $d - 2r - 2$ in the g_i . This means that there are $\deg \mathcal{P} = (d - 1)/2 + 1$ independent integrals, of which the last one is $I_{(d-1)/2} = \mathcal{C}$, the Casimir of the bracket $\{, \}_1$, as given in (3.7).

When d is even, the leading ($v = 0$) part of I_r has the same structure, with terms of each even degree $d - 2r$ appearing for $r = 0, \dots, d/2 - 1$, but for all v the last one is trivial: $I_{d/2} = 2$. Thus, the coefficients of \mathcal{P} provide only $d/2$ independent first integrals. The last non-trivial coefficient is a Casimir of $\{, \}_1$, given by $I_{d/2-1} = \mathcal{C}_1 \mathcal{C}_2 - v$, in terms of the two Casimirs in (3.8), and the Casimir $\mathcal{C}_1 + \mathcal{C}_2$ provides one extra first integral, as required.

In §4c, we outline another proof of theorem 4.1, based on the bi-Hamiltonian structure obtained from the two Poisson brackets $\{, \}_1$ and $\{, \}_2$.

(c) Proof via the bi-Hamiltonian structure

As an alternative to the direct calculation of brackets between the first integrals (4.11), the two Poisson brackets can be used to show complete integrability, with a suitable Lenard–Magri chain; this is a standard method for bi-Hamiltonian systems [37]. This approach applies immediately to the coordinates g_i when d is odd, and the brackets are given by (3.6) and (3.23); but minor modifications are required to apply it to the coordinates f_i when d is even.

For d odd, the first observation to make is that, just as in the case of the dressing chain ($v = 0$), the brackets $\{, \}_1$ and $\{, \}_2$ are compatible with each other, in the sense that their sum, and hence any linear combination of them, is also a Poisson bracket. Thus, these two compatible brackets form a bi-Hamiltonian structure for the KdV map φ . A pencil of Poisson brackets is defined by the bivector field $\mathcal{P}_2 - \lambda \mathcal{P}_1$, where \mathcal{P}_j denotes the bivector corresponding to $\{, \}_j$ for $j = 1, 2$. (In fact, because it defines a bracket for all λ and v , this gives *three* compatible Poisson structures.) Then, by a minor adaptation of theorem 4.8 in Fordy & Hone [22] (which, up to rescaling by a factor of 2, is the special case $\lambda = -1$ and $v = 0$), we see that the trace of the monodromy matrix is a Casimir of the Poisson pencil, or in other words

$$(\mathcal{P}_2 - \lambda \mathcal{P}_1) \lrcorner d\mathcal{P}(\lambda) = 0. \quad (4.13)$$

Expanding this identity in powers of λ yields a finite Lenard–Magri chain, starting with $I_0 = \mathcal{C}^*$, the Casimir of the bracket $\{, \}_2$ (as in remark 3.15), and ending with $I_{(d-1)/2} = \mathcal{C}$, the Casimir of the bracket $\{, \}_1$ (as in lemma 3.5): $\mathcal{P}_2 \lrcorner dI_0 = 0, \mathcal{P}_2 \lrcorner dI_1 = \mathcal{P}_1 \lrcorner dI_0, \dots, \mathcal{P}_2 \lrcorner dI_r = \mathcal{P}_1 \lrcorner dI_{r-1}, \dots, \mathcal{P}_1 \lrcorner dI_{(d-1)/2} = 0$. It then follows, by a standard inductive argument, that $\{I_r, I_s\}_1 = 0 = \{I_r, I_s\}_2$ for $0 \leq r, s \leq (d - 1)/2$. Hence, we see that proposition 4.5 is a consequence of the bi-Hamiltonian structure in this case.

In the case where d is even, a slightly more indirect argument is necessary, making the projection π and working with the map φ' in dimension $d - 1$, as given by (3.26). The main ingredient required is the expression for the relations between the coordinates f_i for $i = 0, \dots, d - 2$ with respect to the first bracket. The case $d = 4$ is special, so we do this example first before summarizing the general case.

Example 4.6. For $d = 4$, we use lemma 3.5 to calculate

$$\begin{aligned} \{f_0, f_1\}_1 &= \{g_0 g_1, g_1 g_2\}_1 = \{g_0, g_1\}_1 g_1 g_2 + g_1 (\{g_0, g_2\}_1 g_1 + g_0 \{g_1, g_2\}_1) \\ &= -g_1 g_2 - g_0 g_1 = -f_0 - f_1, \end{aligned}$$

and similar calculations show that

$$\{f_0, f_2\}_1 = f_1 - \frac{f_0 f_2}{f_1} + v \frac{f_1}{f_0}, \quad \{f_1, f_2\}_1 = -f_1 - f_2 + v \frac{f_1^2}{f_0^2},$$

which implies that f_0, f_1, f_2 generate a three-dimensional Poisson subalgebra for the bracket $\{, \}_1$. It follows that (3.26) is a Poisson map (in three dimensions) with respect to the restriction of this bracket to the subalgebra. Moreover, the restricted bracket $\{, \}_1$ for the f_i is compatible with the

bracket $\{, \}_2$ given by (3.27) with $d = 4$. The coefficients of the corresponding spectral curve, as in example 4.4, can be written as functions of the f_i , i.e.

$$I_0 = f_0 f_2, \quad I_1 = f_0 + f_1 + f_2 + \frac{f_0 f_2}{f_1} + v \frac{f_1}{f_0},$$

and these generate the short Lenard–Magri chain $\{\cdot, I_0\}_2 = 0$, $\{\cdot, I_1\}_2 = \{\cdot, I_0\}_1$, $\{\cdot, I_1\}_1 = 0$.

Theorem 4.7. *For all even $d \geq 6$, the first Poisson structure for the KdV map (3.2) reduces to a bracket for the coordinates $f_i = g_i g_{i+1}$, as specified by the following relations for $v = 1$ and $0 \leq i < j \leq d - 2$:*

$$\{f_i, f_j\}_1 = \begin{cases} -f_i - f_{i+1}, & \text{if } j - i = 1, \\ -\frac{f_i f_{i+2}}{f_{i+1}}, & \text{if } j - i = 2, \\ \frac{f_1 f_3 \dots f_{d-3}}{f_2 f_4 \dots f_{d-4}} \left(1 + \frac{v}{f_0}\right), & \text{if } j - i = d - 2, \\ v \frac{f_1^2 f_3 \dots f_{d-3}}{f_0^2 f_2 \dots f_{d-4}}, & \text{if } i = 1, j = d - 2, \\ 0, & \text{otherwise.} \end{cases} \quad (4.14)$$

This Poisson bracket is preserved by the map (3.26).

The rest of the argument proceeds as for d odd: the brackets $\{, \}_1$ and $\{, \}_2$ (as given in (4.14) and (3.27), respectively) are compatible with each other, so they provide a bi-Hamiltonian structure for φ' in dimension $d - 1$. Letting P'_j for $j = 1, 2$ denote the corresponding Poisson bivector fields, the analogue of (4.13) can then be verified: $(P'_2 - \lambda P'_1)_d \mathcal{P}(\lambda) = 0$. It should be noted that the latter identity is well-defined in the $(d - 1)$ -dimensional space with coordinates f_i : the trace of the monodromy matrix is invariant under the scaling symmetry (3.25), hence all of the first integrals I_r can be written as functions of the variables f_i . In this case, the Lenard–Magri chain begins with $I_0 = C^*$, the Casimir of $\{, \}_2$ (as in remark 3.17), and ends with a Casimir of the bracket $\{, \}_1$, namely $I_{d/2-1} = C_1 C_2 - v$ (which is well-defined in terms of f_i). Thus, it follows that the integrals I_r are in involution with respect to both brackets for the f_i , and this result extends to the bracket $\{, \}_1$ when it is lifted to d dimensions (in terms of g_i , or equivalently v_i).

5. First integrals and integrability of the double potential Korteweg–de Vries maps

Here, we go back to the $(d - 1, -1)$ -reduction of the double pKdV equation (1.3). This reduction yields the difference equation (2.7), and in §2 we showed how writing this as the discrete Euler–Lagrange equation (2.6) led to a symplectic structure for the corresponding $2d$ -dimensional map (2.8). It is easy to see that the first integrals (4.11) of the KdV map also provide first integrals for equation (2.7) by writing $v_i = u_{i-d} - u_i$ for $i = 0, \dots, d - 1$. However, the total number of independent integrals for (3.2) is only $\lfloor d/2 \rfloor + 1$, which is not enough for complete integrability of the $2d$ -dimensional symplectic map (2.8). Here, we will show how to construct sufficiently many additional integrals, leading to a proof of the following result.

Theorem 5.1. *For all $d \geq 3$, the $2d$ -dimensional map given by (2.8) with (2.7) is completely integrable in the Liouville sense.*

The proof of the above, given in §5a, relies on the observation that equation (2.7) can be rewritten as

$$u_{n+1} - u_{n+d-1} - \frac{1}{u_n - u_{n+d}} = u_{n-d+1} - u_{n-1} - \frac{1}{u_{n-d} - u_n},$$

from which we see that

$$u_{n-d+1} - u_{n-1} - \frac{1}{u_{n-d} - u_n} = J_n, \quad \text{with } J_{n+d} = J_n. \quad (5.1)$$

Thus, the function J_n is a d -integral (in the sense of Haggar *et al.* [25]) for the double pKdV map (2.8): for any n , it can be viewed as a function on the $2d$ -dimensional phase space with coordinates (u_{-d}, \dots, u_{d-1}) , and under shifting n it is periodic with period d . This implies that any cyclically symmetric function of J_0, J_1, \dots, J_{d-1} is a first integral for equation (2.7). As we shall see, for the corresponding map (2.8), this has the further consequence that it is *superintegrable*, in the sense that it has more than the number of independent integrals required for Liouville's theorem.

Remark 5.2. For the $(d-1, -1)$ -reduction, the observation that (5.1) holds can be seen as a direct consequence of the fact that (1.3) can be rewritten as (2.4). An analogous observation applies to the (s_1, s_2) -reduction of (1.3), where $n = -s_2\ell + s_1 m$ for coprime s_1, s_2 and in (2.4) one has $J \rightarrow J_n$ with $J_{n+s_1-s_2} = J_n$.

(a) Construction of integrals in involution

In order to construct additional integrals, we need to calculate the Poisson brackets between the J_i as well as their brackets with the v_j ; together, these generate a Poisson subalgebra for the bracket in theorem 2.1, as described by the lemmas 5.3 and 5.4.

Lemma 5.3. *With respect to the non-degenerate Poisson bracket specified by (2.9), the quantities J_i defined in (5.1) and $v_j = u_{j-d} - u_j$ Poisson commute, i.e.*

$$\{J_i, v_j\} = 0, \quad 0 \leq i, j \leq d-1. \quad (5.2)$$

Proof. For $0 < j < d-1$, we have

$$\begin{aligned} \{J_0, v_j\} &= \{u_{-d+1} - u_{-1} - v_0^{-1}, v_j\} = \{u_{-d+1} - u_{-1}, u_{j-d} - u_j\} + v_0^{-2} \{v_0, v_j\} \\ &= -\{u_{-d+1}, u_j\} + v_0^{-2} (-1)^j \prod_{r=0}^j v_r^2 = 0, \end{aligned}$$

where we have used (3.3) as well as theorem 2.1. Similar calculations show that $\{J_0, v_0\} = 0 = \{J_0, v_{d-1}\}$, and, because the bracket is preserved by the double pKdV map (2.8), the vanishing of all the other brackets $\{J_i, v_j\}$ follows by shifting indices. ■

Lemma 5.4. *The Poisson brackets between the d -integrals J_n are specified by*

$$\{J_i, J_j\} = \begin{cases} 1, & \text{if } j - i = 1, \\ -1, & \text{if } j - i = d - 1, \\ 0, & \text{otherwise,} \end{cases} \quad (5.3)$$

for $0 \leq i < j \leq d-1$.

Proof. Once again, from the behaviour under shifting indices, it is enough to verify the brackets for $i = 0$ and $1 \leq j \leq d-1$. Expanding the left-hand side of (5.3), and using lemma 5.3, we obtain

$$\begin{aligned} \{J_0, J_j\} &= \{J_0, u_{j-d+1} - u_{j-1} - v_j^{-1}\} \\ &= \{u_{-d+1} - u_{-1} - (u_{-d} - u_0)^{-1}, u_{j-d+1} - u_{j-1}\} \\ &= -\{u_{-d+1}, u_{j-1}\} + (u_{-d} - u_0)^{-2} (\{u_{-d}, u_{j-d+1}\} - \{u_{-d}, u_{j-1}\}). \end{aligned}$$

Upon substituting with the non-zero brackets for the coordinates u_i , as in theorem 2.1, the above expression vanishes for $2 \leq j \leq d-2$, and is equal to ± 1 for $j = 1, d-1$, respectively. ■

The brackets (5.3) mean that a suitable set of quadratic functions of the J_i give additional integrals for the equation (2.7).

Proposition 5.5. *The functions*

$$T_s = \sum_{i=0}^{d-1} J_i J_{i+s} \quad (5.4)$$

provide $\lfloor d/2 \rfloor + 1$ independent first integrals for the double $pKdV$ map (2.8). These integrals are in involution with each other, and Poisson commute with the first integrals of the KdV map (3.2).

Proof. With indices read mod d , the quantities T_s are cyclically symmetric functions of the J_i ; in other words, they are invariant under the cyclic permutation $(J_0, \dots, J_{d-1}) \mapsto (J_1, \dots, J_{d-1}, J_0)$, which means that they are first integrals for (2.8). From the periodicity of the d -integrals J_i it is clear that $T_{s+d} = T_s$, and also $T_{d-s} = T_s$. Taking $s = 0, \dots, \lfloor d/2 \rfloor$ yields $\lfloor d/2 \rfloor + 1$ independent functions of J_0, \dots, J_{d-1} , and because the J_i are themselves independent functions of u_{-d}, \dots, u_{d-1} , this implies functional independence of this number of the quantities (5.4). The fact that these quantities Poisson commute with each other is a consequence of lemma 5.4, and the fact that $\partial T_r / \partial J_i = J_{i+r} + J_{i-r}$, which implies that

$$\{T_r, T_s\} = \sum_{i=0}^{d-1} \left(\frac{\partial T_r}{\partial J_i} \frac{\partial T_s}{\partial J_{i+1}} - \frac{\partial T_r}{\partial J_{i+1}} \frac{\partial T_s}{\partial J_i} \right) = 0,$$

as required. Also, by lemma 5.3, each T_s Poisson commutes with any function of the v_i , so with the first integrals of (3.2) in particular. ■

Before describing the general case, we now demonstrate complete integrability for the simplest examples.

Example 5.6. Starting with $d = 3$, equation (2.7) yields the six-dimensional symplectic map

$$(u_{-3}, u_{-2}, \dots, u_2) \mapsto (u_{-2}, \dots, u_2, F(u_{-3}, u_{-2}, \dots, u_2)), \quad (5.5)$$

where $F(u_{-3}, u_{-2}, \dots, u_2) = u_0 - (u_1 + u_{-1} - u_2 - u_{-2} + 1/(u_{-3} - u_0))^{-1}$. Two integrals of this map are given in terms of $v_0 = u_{-3} - u_0$, $v_1 = u_{-2} - u_{-1}$, $v_2 = u_{-1} - u_2$ by

$$I_0 \equiv C^* = -v_1 + \frac{1}{v_0} + \frac{1}{v_2} - \frac{1}{v_0 v_1 v_2}, \quad I_1 \equiv C = v_1 - \frac{1}{v_0} - \frac{1}{v_1} - \frac{1}{v_2}.$$

Apart from these, there is the pair of integrals $T_0 = J_0^2 + J_1^2 + J_2^2$, $T_1 = J_0 J_1 + J_1 J_2 + J_2 J_0$, which are written as symmetric functions of the 3-integrals $J_i = u_{i-2} - u_{i-1} - (u_{i-3} - u_i)^{-1}$, $i = 0, 1, 2$. The quantities I_0, I_1, T_0, T_1 Poisson commute with each other, but they cannot all be independent. Indeed, the quantities $v_0, v_1, v_2, J_0, J_1, J_2$ are not themselves independent functions of u_i ; they are connected by the relation

$$I_1 = v_1 - \frac{1}{v_0} - \frac{1}{v_1} - \frac{1}{v_2} = J_0 + J_1 + J_2,$$

which implies that the first integrals satisfy $I_1^2 = T_0 + 2T_1$. Subject to the latter relation, the Liouville integrability of the map (5.5) follows by taking any three independent first integrals in involution (I_0, I_1, T_0 , for instance). The existence of an additional independent first integral, namely $S_0 = J_0 J_1 J_2$ (another symmetric function of J_0, J_1, J_2), means that the map is superintegrable, but this integral does not Poisson commute with T_0 or T_1 .

Example 5.7. For $d = 4$, equation (2.7) gives the eight-dimensional map

$$(u_{-4}, u_{-3}, \dots, u_3) \mapsto (u_{-3}, \dots, u_3, G(u_{-4}, u_{-3}, \dots, u_3)), \quad (5.6)$$

where $G(u_{-4}, u_{-3}, \dots, u_3) = u_0 - (u_1 + u_{-1} - u_3 - u_{-3} + 1/(u_{-4} - u_0))^{-1}$. This map is symplectic with respect to the non-degenerate Poisson structure given in theorem 2.1. There are three

independent integrals that come from the KdV map, given by

$$I_0 = \left(\frac{1}{v_0 v_1} - 1 \right) \left(v_1 - \frac{1}{v_2} \right) \left(v_2 - \frac{1}{v_3} \right), \quad I_1 = C_1 C_2 - 1, \quad \mathcal{K} = C_1 + C_2,$$

which are all written in terms of $v_i = u_{i-4} - u_i$ for $i = 0, 1, 2, 3$, using

$$C_1 = v_1 - \frac{1}{v_0} - \frac{1}{v_2}, \quad C_2 = v_2 - \frac{1}{v_1} - \frac{1}{v_3}.$$

The first two of these integrals (I_0 and I_1) are coefficients of the spectral curve in example 4.4 (with $v = 1$), whereas the third is not. As well as these, there are three independent cyclically symmetric quadratic functions of the 4-integrals $J_i = u_{i-3} - u_{i-1} - (u_{i-4} - u_i)^{-1}$, $i = 0, 1, 2, 3$, namely

$$T_0 = J_0^2 + J_1^2 + J_2^2, \quad T_1 = J_0 J_1 + J_1 J_2 + J_2 J_3 + J_3 J_0, \quad T_2 = 2(J_0 J_2 + J_1 J_3),$$

which are also first integrals of (5.6). There are two relations between the quantities v_0, \dots, v_3 and J_0, \dots, J_3 , as can be seen by noting the identities $C_1 = J_0 + J_2$, $C_2 = J_1 + J_3$; thus, the aforementioned first integrals are related by $\mathcal{K}^2 = T_0 + 2T_1 + T_2$, $I_1 + 1 = T_1$. Hence, complete integrability of the map (5.6) follows from the existence of four independent integrals in involution, i.e. $I_0, I_1, \mathcal{K}, T_0$. The map is also superintegrable, owing to the presence of a fifth independent first integral, given by another symmetric function of the J_i , that is $S_0 = J_0 J_1 J_2 J_3$.

As we now briefly explain, the general case follows the pattern of one of the preceding two examples very closely, according to whether d is odd or even.

When d is odd, the spectral curve (4.7) has $(d+1)/2$ non-trivial coefficients, which are the quantities I_r appearing in (4.10). There are also $(d+1)/2$ independent functions T_s , as in (5.4), but the identity $I_{(d-1)/2} = J_0 + J_1 + \dots + J_{d-1}$ implies that these two sets of functions are related by $I_{(d-1)/2}^2 = \sum_{s=0}^{d-1} T_s$. Hence, there are precisely d independent functions in involution, as required for theorem 5.1.

In the case that d is even, the non-trivial coefficients I_r of the spectral curve are supplemented by the additional integral $\mathcal{K} = C_1 + C_2$, providing a total of $(d+2)/2$ independent functions, and there are the same number of independent functions of the form (5.4), but now the pair of identities

$$C_1 = \sum_{i \text{ even}, 0 \leq i \leq d-2} J_i, \quad C_2 = \sum_{i \text{ odd}, 1 \leq i \leq d-1} J_i$$

together imply that the first integrals satisfy the two relations $\mathcal{K}^2 = \sum_{s=0}^{d-1} T_s$, $I_{(d-2)/2} + 1 = \sum_{s \text{ odd}, 1 \leq s \leq d/2} T_s$, so once again, there are d independent integrals, as required.

One can also construct extra first integrals S_j for $j = 0, \dots, \lfloor (d-1)/2 \rfloor - 1$, by taking additional independent cyclically symmetric functions of the J_i . This means that the map (2.8) is superintegrable for all d .

(b) Difference equations with periodic coefficients

To conclude this section, we look at equation (5.1) in a different way, and show how it is related to other difference equations with periodic coefficients. We begin by revisiting the previous two examples.

Example 5.8. For the map (5.5), the introduction of the variables $x_n = u_n - u_{n+1}$ yields the following difference equation of second order:

$$x_{n+2} + x_n = \frac{1}{x_{n+1} - J_n} - x_{n+1}, \quad J_{n+3} = J_n. \quad (5.7)$$

Iteration of the latter preserves the canonical symplectic form $dx_0 \wedge dx_1$ in the (x_0, x_1) plane. Apart from the coefficient J_n , there is another periodic quantity associated with this equation, namely

the 3-integral H_n which equals

$$\begin{aligned} & x_n^2 x_{n+1} + x_{n+1}^2 x_n + (J_{n+2} J_n - J_{n+1} J_n - 1) x_n + (J_{n+2} J_n - J_{n+1} J_{n+2} - 1) x_{n+1} \\ & - J_n x_n^2 - J_{n+2} x_{n+1}^2 + (J_{n+1} - J_n - J_{n+2}) x_n x_{n+1} \end{aligned}$$

(with $H_{n+3} = H_n$). This is related to some of the first integrals for $d = 3$ by the identity $H_n - J_{n+1} = I_0 - S_0$, where $S_0 = J_0 J_1 J_2$ as in example 5.6.

Example 5.9. By introducing $w_n = u_n - u_{n+2}$ in (5.6), we obtain the second-order equation

$$w_{n+2} + w_n = \frac{1}{w_{n+1} - J_n}, \quad J_{n+4} = J_n. \quad (5.8)$$

Each iteration of this difference equation preserves the canonical symplectic structure $dw_0 \wedge dw_1$. Aside from the coefficient J_n , the equation (5.8) has a 4-integral, i.e. $H_{n+4} = H_n$, where H_n is given by

$$\begin{aligned} & w_n^2 w_{n+1}^2 + (J_{n+2} - J_n) w_n^2 w_{n+1} + (J_{n+1} - J_{n+3}) w_n w_{n+1}^2 \\ & - J_n J_{n+2} w_n^2 - J_{n+1} J_{n+3} w_{n+1}^2 + ((J_{n+3} - J_{n+1})(J_n - J_{n+2}) - 1) w_n w_{n+1} \\ & + (J_n J_{n+1} J_{n+3} - J_{n+1} J_{n+2} J_{n+3} - J_{n+1}) w_{n+1} \\ & + (J_n J_{n+2} J_{n+3} - J_n J_{n+1} J_{n+2} - J_{n+2}) w_n. \end{aligned}$$

One can check that $H_{n+1} - H_n = J_{n+2}(J_{n+3} - J_{n+1})$, and H_n is related to the first integrals in example 5.7 by the formula $H_n - J_{n+1} J_{n+2} = I_0 - S_0$.

Remark 5.10. Both equations (5.7) and (5.8) are of the type considered recently by Roberts & Jogle [35]: their orbits move periodically through a sequence of biquadratic curves, defined by the quantities H_n , and they reduce to symmetric QRT maps when $J_n = \text{constant}$.

In general, equation (5.1) can be rewritten in terms of the variables $x_n = u_{n-d} - u_{n-d+1}$, to obtain a difference equation of order $d - 1$, that is

$$x_n + x_{n+1} + \cdots + x_{n+d-1} = \frac{1}{x_{n+1} + x_{n+2} + \cdots + x_{n+d-2} - J_n}, \quad (5.9)$$

where J_n is periodic with period d . Note that both sides of equation (5.9) are equal to v_n , the discrete KdV variable, as in (3.1). This equation can be seen as a higher-order analogue of the McMillan map, with periodic coefficients. First integrals for equation (5.9) can be obtained from the integrals I_r of the KdV map (3.2) by rewriting all the variables v_i (or g_i) in terms of x_n and J_n . To be precise, one can verify that $g_i = x_{i-1} + x_i + J_i$ for $1 \leq i \leq d - 1$, and $v_0 = -g_0^{-1}$ is given by the right-hand side of (5.9) for $n = 0$. In particular, the explicit formula for I_0 is

$$I_0 = \prod_{i=1}^{d-2} (x_{i-1} + x_i + J_i) \left(\left(\sum_{i=1}^{d-2} x_i - J_0 \right) \left(\sum_{i=0}^{d-3} x_i - J_{d-1} \right) - 1 \right).$$

When d is even, one can reduce (5.9) to a difference equation of order $d - 2$, by setting $w_n = x_n + x_{n+1}$. Formulae for first integrals in terms of the variables w_n follow directly from integrals given in terms of x_n and J_n .

6. Conclusions

We have demonstrated that the $(d - 1, -1)$ -reduction of the discrete KdV equation (1.4) is a completely integrable map in the Liouville sense. There are two different Poisson structures for this map: one was obtained by starting from the related double pKdV equation (1.3) and its associated Lagrangian; the other arose by using tau-functions and a connection with cluster algebras. The appropriate reduction of the Lax pair (4.2) for discrete KdV, via the staircase method, was the key to finding explicit expressions for first integrals, and two ways were presented to

prove that these are in involution. The corresponding reduction of the lattice equation (1.3) was also seen to be completely integrable (and even superintegrable), with additional first integrals appearing owing to the presence of the d -integral J_n .

An interesting feature of all these reductions is that, although they are autonomous difference equations, they have various difference equations with periodic coefficients associated with them, such as (3.19), (3.20) and (5.9).

There are several ways in which the results in this paper could be developed further. In particular, it would be interesting to understand the Poisson brackets for the reductions in terms of an appropriate r -matrix. It would also be instructive to make use of the bi-Hamiltonian structure to perform separation of variables, by the method in Błaszak [38], and this could be further used to obtain the exact solution of these difference equations in terms of theta functions associated with the hyperelliptic spectral curve (4.7). Finally, it would be worthwhile to extend the results here to (s_1, s_2) -reductions of discrete KdV, and to reductions of other integrable lattice equations that have not been considered before.

This work was supported by the Australian Research Council. D.T.T. visited the University of Kent in 2011 and 2012, and is grateful for the support of an Edgar Smith Scholarship which funded her travel. A.N.W.H. thanks the organizers of the Nonlinear Dynamical Systems workshop for supporting his trip to La Trobe University, Melbourne in September–October 2012.

References

- Zakharov VE (ed.) 1991 *What is integrability?* Springer Series in Nonlinear Dynamics. Berlin, Berlin: Springer.
- Mikhailov AV (ed.) 2009 *Integrability*. Springer Lecture Notes in Physics, no. 767. Berlin, Heidelberg: Springer.
- Quispel GRW, Roberts JAG, Thompson CJ. 1988 Integrable mappings and soliton equations. *Phys. Lett. A* **126**, 419–421. (doi:10.1016/0375-9601(88)90803-1)
- Capel HW, Nijhoff FW, Papageorgiou VG. 1991 Complete integrability of Lagrangian mappings and lattices of KdV type. *Phys. Lett. A* **155**, 377–387. (doi:10.1016/0375-9601(91)91043-D)
- Ramani A, Grammaticos B. 1996 Discrete Painlevé equations: coalescences, limits and degeneracies. *Physica A* **228**, 160–171. (doi:10.1016/0378-4371(95)00439-4)
- Nijhoff FW, Capel HW. 1995 The discrete Korteweg–de Vries equation. *Acta Appl. Math.* **39**, 133–158. (doi:10.1007/BF00994631)
- Adler VE, Bobenko AI, Suris YB. 2003 Classification of integrable equations on quad-graphs the consistency approach. *Commun. Math. Phys.* **233**, 513–543.
- Suris YB. 2003 *The problem of integrable discretization: Hamiltonian approach*. Progress in Mathematics **219**, Basel, Switzerland: Birkhauser.
- Krichever I, Lipan O, Wiegmann P, Zabrodin A. 1997 Quantum integrable models and discrete classical Hirota equations. *Commun. Math. Phys.* **188**, 267–304. (doi:10.1007/s002200050165)
- Maeda S. 1987 Completely integrable symplectic mapping. *Proc. Japan Acad. A* **63**, 198–200. (doi:10.3792/pjaa.63.198)
- Bruschi M, Ragnisco O, Santini PM, Gui-Zhang T. 1991 Integrable symplectic maps. *Physica D* **49**, 273–294. (doi:10.1016/0167-2789(91)90149-4)
- Veselov AP. 1991 Integrable maps. *Russ. Math. Surveys* **46**, 1–51. (doi:10.1070/RM1991v046n05ABEH002856)
- Quispel GRW, Capel HW, Papageorgiou VG, Nijhoff FW. 1991 Integrable mappings derived from soliton equations. *Physica A* **173**, 243–266. (doi:10.1016/0378-4371(91)90258-E)
- van der Kamp PH. 2009 Initial value problems for lattice equations. *J. Phys. A, Math. Theor.* **42**, 404019. (doi:10.1088/1751-8113/42/40/404019)
- van der Kamp PH, Quispel GRW. 2010 The staircase method: integrals for periodic reductions of integrable lattice equations. *J. Phys. A, Math. Theor.* **43**, 465207. (doi:10.1088/1751-8113/43/46/465207)
- Tran DT, van der Kamp PH, Quispel GRW. 2009 Closed-form expressions for integrals of traveling wave reductions of integrable lattice equations. *J. Phys. A, Math. Theor.* **42**, 225201. (doi:10.1088/1751-8113/42/22/225201)

17. van der Kamp PH, Rojas O, Quispel GRW. 2007 Closed-form expressions for integrals of mKdV and sine-Gordon maps. *J. Phys. A, Math. Theor.* **39**, 12789. (doi:10.1088/1751-8113/40/42/S21)
18. Tran DT. 2011 Complete integrability of maps obtained as reductions of integrable lattice equations. PhD thesis, La Trobe University, Australia.
19. Tran DT, van der Kamp PH, Quispel GRW. 2011 Involutivity of integrals of sine-Gordon, modified KdV and potential KdV maps *J. Phys. A, Math. Theor.* **44**, 295206. (doi:10.1088/1751-8113/44/29/295206)
20. Gekhtman M, Shapiro M, Vainshtein A. 2005 Cluster algebras and Weil–Petersson forms. *Duke Math. J.* **127**, 291–311. (doi:10.1215/S0012-7094-04-12723-X)
21. Fordy AP, Hone ANW. 2011 Symplectic maps from cluster algebras. *SIGMA* **7**, 091. (doi:10.3842/SIGMA.2011.091)
22. Fordy AP, Hone ANW. Discrete integrable systems and Poisson algebras from cluster maps. (<http://arxiv.org/abs/1207.6072>)
23. Quispel GRW, Nijhoff FW, Capel HW, van der Linden J. 1984 Linear integral equations and nonlinear difference–difference equations. *Physica A* **125**, 344–380. (doi:10.1016/0378-4371(84)90059-1)
24. Wiersma GL, Capel HW. 1987 Lattice equations, hierarchies and Hamiltonian structures. *Physica A* **142**, 199–244. (doi:10.1016/0378-4371(87)90024-0)
25. Haggar FA, Byrnes GB, Quispel GRW, Capel HW. 1996 k -integrals and k -Lie symmetries in discrete dynamical systems. *Physica A* **233**, 379–394. (doi:10.1016/S0378-4371(96)00142-2)
26. Błaszak M. 1998 *Multi-hamiltonian theory of dynamical systems*. Berlin, Germany: Springer.
27. Veselov AP, Shabat AB. 1993 Dressing chains and the spectral theory of the Schrödinger operator. *Funct. Anal. Appl.* **27**, 81–96. (doi:10.1007/BF01085979)
28. Hirota R. 1977 Nonlinear partial difference equations. I. Difference analogue of the Korteweg–de Vries equation. *J. Phys. Soc. Japan* **43**, 1424–1433. (doi:10.1143/JPSJ.43.1424)
29. Somos M. 1989 Problem 1470. *Crux Math.* **15**, 208.
30. Gale D. 1991 The strange and surprising saga of the Somos sequences. *Math. Intell.* **13**, 40–42. (doi:10.1007/BF03024070)
31. Robinson R. 1992 Periodicity of Somos sequences. *Proc. Amer. Math. Soc.* **116**, 613–619. (doi:10.1090/S0002-9939-1992-1140672-5)
32. Fomin S, Zelevinsky A. 2002 The Laurent phenomenon. *Adv. Appl. Math.* **28**, 119–144. (doi:10.1006/aama.2001.0770)
33. Fordy AP, Marsh RJ. 2011 Cluster mutation-periodic quivers and associated Laurent sequences. *J. Algebr. Comb.* **34**, 19–66. (doi:10.1007/s10801-010-0262-4)
34. Fomin S, Zelevinsky A. 2002 Cluster algebras I: foundations. *J. Am. Math. Soc.* **15**, 497–529. (doi:10.1090/S0894-0347-01-00385-X)
35. Roberts JAG, Jorgia D. 2013 Birational maps that send biquadratic curves to biquadratic curves. Preprint.
36. Nijhoff FW. 2006 *Discrete systems and integrability*. Lecture Notes in Mathematics 3490. University of Leeds, UK.
37. Magri F. 1978 A simple model of an integrable Hamiltonian equation, *J. Math. Phys.* **19**, 1156–1162. (doi:10.1063/1.523777)
38. Błaszak M. 2002 From bi-Hamiltonian geometry to separation of variables: stationary Harry–Dym and the KdV dressing chain. *J. Nonlinear Math. Phys.* **9**(Suppl. 1), 1–13. (doi:10.2991/jnmp.2002.9.s1.1)