

HOMOGENEOUS DARBOUX POLYNOMIALS AND GENERALISING INTEGRABLE ODE SYSTEMS

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(Communicated by Ben Leimkuhler)

ABSTRACT. We show that any system of ODEs can be modified whilst preserving its homogeneous Darboux polynomials. We employ the result to generalise a hierarchy of integrable Lotka-Volterra systems.

1. **Introduction.** We are concerned with systems of Ordinary Differential Equations (ODEs),

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad (1)$$

where $\dot{\mathbf{x}}$ denotes the time derivative of a vector \mathbf{x} . A Darboux polynomial (or second integral) of (1) is a polynomial $P(\mathbf{x})$ such that $\dot{P} = C(\mathbf{x})P$ for some function C which is called the cofactor of P [7]. Darboux polynomials are important as the existence of sufficiently many Darboux polynomials implies the existence of a first integral, cf. Theorems 2.2 and 2.3 in [7]. Recently their use was extended to the discrete setting in [3].

In this paper, we propose the following generalisation of any ODE system of the form (1):

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + b(\mathbf{x}, t)\mathbf{x}, \quad (2)$$

where b is a scalar function of \mathbf{x}, t . We will prove that if P is a homogeneous Darboux polynomial for (1), then P is also a Darboux polynomial for (2) with a modified cofactor.

We show that in several examples the above generalisation preserves the integrability of the ODE, e.g. this is the case for generalisations of: (i) the 2-dimensional system

$$\begin{aligned} \dot{x} &= x^2 + 2xy + 3y^2, \\ \dot{y} &= 2y(2x + y), \end{aligned} \quad (3)$$

found in [5, Appendix], (ii) the 4-dimensional Lotka-Volterra (LV) system

$$\begin{aligned} \dot{x}_1 &= x_1(+x_2 + x_3 + x_4) \\ \dot{x}_2 &= x_2(-x_1 + x_3 + x_4) \\ \dot{x}_3 &= x_3(-x_1 - x_2 + x_4) \\ \dot{x}_4 &= x_4(-x_1 - x_2 - x_3), \end{aligned} \quad (4)$$

2020 *Mathematics Subject Classification.* 34C14, 70H06.

Key words and phrases. Darboux polynomials, Lotka-Volterra systems, Liouville integrability, superintegrability, ordinary differential equations.

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as well as (iii) higher dimensional LV systems found in [10]. For the LV systems we show that both Liouville integrability and superintegrability are preserved under certain generalisations given by (2).

2. Darboux polynomials and integrals/integrability. Note that if P_1 and P_2 are Darboux polynomials with cofactors C_1 and C_2 respectively, the product $P_1^a P_2^b$ is a Darboux polynomial with cofactor $aC_1 + bC_2$. This implies that linear relations between cofactors give rise to integrals.

For the 2-dimensional system (3) three Darboux polynomials

$$P_1 = x + y, \quad P_2 = x - y, \quad P_3 = y, \quad (5)$$

with cofactors given by

$$C_1 = x + 5y, \quad C_2 = x - y, \quad C_3 = 4x + 2y, \quad (6)$$

respectively, were given in [7, Example 2.21]. As these cofactors satisfy the linear relation $C_1 + 3C_2 - C_3 = 0$, an integral is given by

$$I = P_1 P_2^3 P_3^{-1} = \frac{(x+y)(x-y)^3}{y}.$$

The 4-dimensional LV system (4) admits linear Darboux polynomials of the form

$$P_{i,j} = \sum_{k=i}^j x_k, \quad \text{with } 1 \leq i \leq j \leq 4,$$

with corresponding cofactor

$$C_{i,j} = -\sum_{k=1}^{i-1} x_k + \sum_{k=j+1}^n x_k.$$

Because

$$C_{1,2} - C_{3,3} + C_{4,4} = (x_3 + x_4) - (-x_1 - x_2 + x_4) + (-x_1 - x_2 - x_3) = 0,$$

the rational function

$$F = P_{1,2} P_{3,3}^{-1} P_{4,4} = (x_1 + x_2) \frac{x_4}{x_3}$$

is an integral. And similarly,

$$C_{3,4} - C_{2,2} + C_{1,1} = (-x_1 - x_2) - (-x_1 + x_3 + x_4) + (x_2 + x_3 + x_4) = 0$$

yields the rational integral

$$G = P_{3,4} P_{2,2}^{-1} P_{1,1} = (x_3 + x_4) \frac{x_1}{x_2}.$$

As $C_{1,4} = 0$, the function

$$H = P_{1,4} = x_1 + x_2 + x_3 + x_4$$

provides a third integral. The functions F, G, H are functionally independent, as their gradients are linearly independent, and therefore the LV system (4) is superintegrable. The variables $u_i = P_{1,i}$ provide a separation of variables, i.e. each variable satisfies the same differential equation $\dot{u}_i = u_i(H - u_i)$ which can be explicitly integrated, cf. [2]

The system (4) is also a Hamiltonian system, with Hamiltonian H and quadratic Poisson bracket, of rank 4,

$$\{x_i, x_j\} = x_i x_j, \quad i < j. \quad (7)$$

As both F and G Poisson commute with H , the systems F, H and G, H , and hence the vector field (4), are Liouville integrable, cf. [10].

3. Generalising ODE systems. The following result is quite general, it generalises any ODE system (1) whilst preserving all homogeneous Darboux polynomials.

Theorem 3.1. *Let $P(\mathbf{x})$ be a homogeneous Darboux polynomial of degree d with cofactor $C(\mathbf{x})$ for the system of ODEs $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. Then P is a Darboux polynomial for the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + b(\mathbf{x}, t)\mathbf{x}$, with cofactor $C(\mathbf{x}) + db(\mathbf{x}, t)$, where b is a scalar function of \mathbf{x}, t .*

Proof. As P is homogeneous of degree d , we have $\mathbf{x} \cdot \nabla P = dP$. As P is a Darboux polynomial for $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, we have $\dot{P} = \nabla P \cdot \mathbf{f} = CP$. For the generalised system we then have

$$\dot{P} = \nabla P \cdot (\mathbf{f} + b\mathbf{x}) = CP + bdP = (C + db)P. \quad \square$$

We first apply Theorem 3.1 to the 2-dimensional system (3). With $b = ax + cy$ we obtain a generalisation of (3),

$$\begin{aligned} \dot{x} &= x^2 + 2xy + 3y^2 + (ax + cy)x, \\ \dot{y} &= 2y(2x + y) + (ax + cy)y. \end{aligned} \quad (8)$$

Each P_i , $i = 1, 2, 3$, given by (5), is a linear Darboux polynomial for the system (8) with modified cofactor $C'_i = C_i + ax + cy$, where C_i is given by (6). As

$$(c - a - 2)C'_1 - (a + c + 6)C'_2 + 2(a + 1)C'_3 = 0,$$

the function

$$K = P_1^{c-a-2} P_2^{-(a+c+6)} P_3^{2(a+1)} = \frac{(x+y)^{c-a-2} y^{2(a+1)}}{(x-y)^{a+c+6}}.$$

is a first integral of (8).

Applying Theorem 3.1 to the 4-dimensional system (4), taking b to be a constant, yields

$$\begin{aligned} \dot{x}_1 &= x_1(b + x_2 + x_3 + x_4) \\ \dot{x}_2 &= x_2(b - x_1 + x_3 + x_4) \\ \dot{x}_3 &= x_3(b - x_1 - x_2 + x_4) \\ \dot{x}_4 &= x_4(b - x_1 - x_2 - x_3), \end{aligned} \quad (9)$$

whose Darboux polynomials $P_{i,j}$ now have cofactors $C'_{i,j} = C_{i,j} + b$. In particular, $H = P_{1,4}$ is no longer an integral, and the linear combinations $C'_{1,2} - C'_{3,3} + C'_{4,4} = C'_{3,4} - C'_{2,2} + C'_{1,1} = b$ do not vanish. We have to subtract the cofactor $C'_{1,4} = b$, which corresponds to dividing by H . This yields two integrals

$$F' = \frac{(x_1 + x_2)x_4}{(x_1 + x_2 + x_3 + x_4)x_3},$$

and

$$G' = \frac{(x_3 + x_4)x_2}{(x_1 + x_2 + x_3 + x_4)x_1}.$$

The new system (9) is still Hamiltonian, with the same bracket (7). The new Hamiltonian

$$H' = H - b \ln \left(\frac{x_1 x_3}{x_2 x_4} \right)$$

is no longer rational. The integrals F', G', H' are functionally independent, and so the system (9) is superintegrable. Moreover, the functions F' and G' Poisson commute with H' , hence the systems F', H' and G', H' are Liouville integrable. In the next section we generalise this example to arbitrary even dimensions.

4. Integrability of a generalised n -dimensional LV system. In [10] the system of ODEs

$$\dot{x}_i = x_i \left(\sum_{j>i} x_j - \sum_{j<i} x_j \right), \quad i = 1, \dots, n, \quad (10)$$

arose as a subsystem of the quadratic vector fields associated with multi-sums of products, and it was shown to be superintegrable as well as Liouville integrable. Integrable generalisations of the system (10) have been obtained in [4, 6, 8]. The generalisation

$$\dot{x}_i = x_i \left(b + \sum_{j>i} x_j - \sum_{j<i} x_j \right), \quad i = 1, \dots, n, \quad (11)$$

of which (9) is a special case, seems to be new. In [10] the LV system of ODEs (10), with $n = 2r$ even, was shown to admit the integrals, for $k = 1, \dots, r$,

$$\begin{aligned} F_k &= (x_1 + x_2 + \dots + x_{2k}) \frac{x_{2k+2} x_{2k+4} \dots x_n}{x_{2k+1} x_{2k+3} \dots x_{n-1}}, \\ G_k &= (x_{n-2k+1} + x_{n-2k+2} + \dots + x_n) \frac{x_1 x_3 \dots x_{n-2k-1}}{x_2 x_4 \dots x_{n-2k}}, \end{aligned} \quad (12)$$

The $n - 1$ integrals $F_1, \dots, F_{r-1}, G_1, \dots, G_{r-1}, F_r = G_r = H = P_{1,n}$ were proven to be independent, and the sets $\{F_1, \dots, F_{r-1}, H\}$, $\{G_1, \dots, G_{r-1}, H\}$ were proven to pairwise Poisson commute with respect to the bracket (7), which has rank n . Similar results were obtained for n odd (here the rank of (7) is $n - 1$), establishing the superintegrability as well as Liouville integrability of the n -dimensional LV system (10) for all n . We consider a generalisation of the even-dimensional system.

Theorem 4.1. *The system*

$$\dot{x}_i = x_i \left(b + \sum_{j>i} x_j - \sum_{j<i} x_j \right), \quad i = 1, \dots, n, \quad (13)$$

where $n = 2r$ is even, is both superintegrable and Liouville integrable.

Proof. The system is Hamiltonian with Hamiltonian

$$H' = H - bS, \quad \text{with } S = \ln \left(\frac{x_1 x_3 \dots x_{n-1}}{x_2 x_4 \dots x_n} \right).$$

According to Theorem 3.1 the functions (12) and H are Darboux functions (functions F such that $\dot{F} = C(\mathbf{x})F$ for some C) with cofactor b . Therefore, $n - 2$ integrals

We define lower and upper triangular matrices

$$L_{i,k} = \begin{cases} M_{i,k} & k = 1, 2 \\ 1 & k = i \\ k/(n-k) & 1 \equiv i = k+1, k > 2 \\ 0 & \text{otherwise,} \end{cases}$$

$$U_{k,j} = \begin{cases} M_{k,j} & k = 1, 2 \\ -n(n-k)/2 & k \equiv 1, j \equiv 1, j \geq k \\ n(n-k+2)/2 & k \equiv 1, j \equiv 0, j \geq k \\ -n^2/(n-k+1) & k \equiv 0, j \equiv 0, j \geq k \\ 0 & k \equiv 0, j \equiv 0, j \geq k \text{ or } k > j. \end{cases}$$

When $n = 10$ we have

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 36 & -44 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 4 & \frac{3}{7} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 27 & -33 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 18 & -22 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 & 0 & \frac{7}{3} & 1 & 0 & 0 \\ 9 & -11 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & 1 \end{pmatrix},$$

$$U = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & -35 & 45 & -35 & 45 & -35 & 45 & -35 & 45 \\ 0 & 0 & 0 & -\frac{100}{7} & 0 & -\frac{100}{7} & 0 & -\frac{100}{7} & 0 & -\frac{100}{7} \\ 0 & 0 & 0 & 0 & -25 & 35 & -25 & 35 & -25 & 35 \\ 0 & 0 & 0 & 0 & 0 & -20 & 0 & -20 & 0 & -20 \\ 0 & 0 & 0 & 0 & 0 & 0 & -15 & 25 & -15 & 25 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{100}{3} & 0 & -\frac{100}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5 & 15 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -100 \end{pmatrix}.$$

We now show that $J = LU$, making use of the Kronecker delta, $\delta_{i,k} = 1$ if $i = k$ and 0 otherwise, and using summation over repeated indices. There are three cases:

- $i = 1, 2$. We have $L_{i,k} = \delta_{i,k}$, so $L_{i,k}U_{k,j} = U_{i,j} = M_{i,j}$.
- $1 \equiv i > 2$. We have

$$L_{i,k}U_{k,j} = (n-1)(n-i+1)U_{1,j}/2 - (n+1)(n-i+1)U_{2,j}/2 + U_{i,j}$$

$$= \begin{cases} -((-1)^j n + 1)(n-i+1)/2 & i > j \\ (n-1)(n-i+1)/2 - n(n-i)/2 = (i-1)/2 & 1 \equiv j \geq i \\ -(n+1)(n-i+1)/2 + n(n-i+2)/2 = (i-1)/2 & 0 \equiv j \geq i. \end{cases}$$

- $0 \equiv i > 2$. We have

$$\begin{aligned} L_{i,k}U_{k,j} &= (n-i+2)(U_{1,j} + U_{2,j})/2 + (i-1)U_{i-1,j}/(n-i+1) + U_{i,j} \\ &= \begin{cases} \frac{n-i+2}{2} & j < i-1 \\ \frac{n-i+2}{2} - \frac{(i-1)n(n-i+1)}{2(n-i+1)} = -\frac{(i-2)(n+1)}{2} & 1 \equiv j \geq i \\ \frac{n-i+2}{2} + \frac{(i-1)n(n-i+3)}{2(n-i+1)} - \frac{n^2}{n-i+1} = \frac{(i-2)(n-1)}{2} & 0 \equiv j \geq i. \end{cases} \end{aligned}$$

As both L and U have non-zero diagonal elements, the determinant of J is non-zero. Hence the set S is functionally independent. This shows that (13) is superintegrable.

Next we prove that each pair of functions in the set $\{F'_1, \dots, F'_{r-1}, H'\}$ Poisson commutes with respect to the bracket (7). Due to the Leibniz rule, the brackets $\{F'_i, F'_j\} = \{F_i/H, F_j/H\}$, with $1 \leq i, j < r$, can be expressed in terms of $\{F_i, F_j\}$, $\{F_i, H\}$, $\{H, F_j\}$, which all vanish. We also have $\{F'_i, H'\} = 0$ as the F'_i are integrals and H' is the Hamiltonian function of the system. Similarly, it follows that the functions in $\{G'_1, \dots, G'_{r-1}, H'\}$ Poisson commute. This shows that (13) is Liouville integrable. \square

Remark 1. Similar to the above, one can also show that the system

$$\dot{x}_i = x_i \left(b(S) + \sum_{j>i} x_j - \sum_{j<i} x_j \right), \quad i = 1, \dots, n = 2r, \quad (14)$$

where b is an arbitrary integrable function, is both superintegrable and Liouville integrable. The system (14) is a Hamiltonian system with Hamiltonian $H^* = H - B(S)$, where B is the anti-derivative of b .

Remark 2. In general, the b -generalisation (2) of a Hamiltonian system (1) will not be Hamiltonian. We hope to discuss some other cases in which the generalisation is Hamiltonian in a future publication. The reason that we have restricted the dimension of the Lotka-Volterra systems (13) to be even is that it seems unclear whether a Hamiltonian exists in the general odd-dimensional case.

Remark 3. Darboux polynomials (special integrals) have been considered in a non-autonomous setting in [1]. Theorem 3.1 remains valid when \mathbf{f} , P and C are non-autonomous and P is homogeneous in \mathbf{x} .

Acknowledgements. The authors thank Pambos Evripidou for careful reading of the manuscript and his suggestions to include a generalisation of (3) as well as Remark 1. GRWQ is grateful to the Simons Foundation for a Fellowship during the early stages of this work, and acknowledges support from the European Union Horizon 2020 research and innovation programmes under the Marie Skłodowska-Curie grant agreement No. 691070.

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Received February 2020; revised May 2020.

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