JOURNAL OF DIFFERENCE EQUATIONS AND APPLICATIONS, 2016
VOL. 22, NO. 6, 789–816
http://dx.doi.org/10.1080/10236198.2016.1142980

Taylor & Francis
Taylor & Francis Group

From discrete integrable equations to Laurent recurrences

Khaled Hamad and Peter H. van der Kamp

Department of Mathematics and Statistics, La Trobe University, Melbourne, Australia

ABSTRACT

We show how to obtain relations for the divisors of terms generated by a homogenized version of a rational recurrence. When the rational recurrence confines singularities the relations take the form of a rational recurrence, possibly with periodic coefficients. As the recurrence generates polynomials one expects it to possess the Laurent property. The method we develop uses ultra-discretization and recursive factorization. It is applied to certain QRT-maps which gives rise to Somos-k ($k = 4, 5$) sequences with periodic coefficients. Novel $(N + 3)$-rd order recurrences are obtained from the Nth order DTKQ-equation ($N = 2, 3$). In each case the resulting recurrence equation has the Laurent property. The method is equally applicable to non-integrable or non-confining equations. However, in the latter case the degree and the order of the relation might display unbounded growth. We demonstrate the difference, by considering different parameter choices in a generalized Lyness equation.

ARTICLE HISTORY
Received 12 August 2015
Accepted 10 January 2016

KEYWORDS
Difference equations; integrability; Laurent recurrences; singularity confinement; ultra-discretization; QRT-map; Somos sequence; generalized Lyness equation

1. Introduction

A sequence $\{u_n\}_{n=1}^{\infty}$ defined by N initial values $\{u_n\}_{n=1}^{N}$ and an Nth order rational recursion,

$$u_{n+N} = R(u_n, u_{n+1}, \dots, u_{n+N-1}), \tag{1}$$

where R is a rational function, is said to have the *Laurent property* if, for all n, u_n is polynomial in the variables $\{u_n^{\pm 1}\}_{n=1}^{N}$. The property was first introduced by Hickerson to prove the integrality of a sequence called Somos-6, cf. [48]. Indeed, as an immediate consequence of the Laurent property it follows that the sequence obtained by taking $\{u_n = 1\}_{n=1}^{N}$ is an integer sequence, or, a sequence of polynomials if the rational function R depends (polynomially) on additional parameters. For example, with mentioned initial values the (generalized) Somos-4 recurrence,

$$\tau_{n+2}\tau_{n-2} = \alpha\tau_{n+1}\tau_{n-1} + \beta\tau_n^2, \tag{2}$$

provides a sequence of polynomials in two variables α, β.

Equation (2) was derived (in 1982) by Michael Somos as an addition formula for elliptic functions. It is the prototype Laurent recurrence, and it has many beautiful properties. The sequence of numbers that one gets by taking $\alpha = \beta = 1$ is referred to as the Somos-4

CONTACT Peter H. van der Kamp P.vanderKamp@latrobe.edu.au

sequence. Its integrality (and of related sequences) was a great mystery initially [19,39,51, 59]. Robinson showed that the ith and jth terms of the Somos-4 sequence are relatively prime whenever $|i-j| \leq 4$, and he inferred that for any given $m \in \mathbb{N}$ the sequence modulo m is periodic [48]. Everest et al. [9] showed that every term beyond the fourth has a primitive divisor, i.e. a prime which does not divide any preceding term. Kanki et al. [36] have proven that all terms of Somos-4 are irreducible Laurent polynomials in their initial values and pairwise co-prime, as Laurent polynomials. A seemingly unnoticed divisibility property for the Somos-4 polynomials, and hence for the Somos-4 sequence, was recently found by one of the authors [33]. A so called near-addition formula has been proven in [38]. Somos-4 is closely connected to an elliptic divisibility sequence [29,32,44,52,56], the theory of which recently found application in cryptography [49], and in generating large primes [10]. An explicit solution for τ_n in terms of the Weierstrass elliptic function can be found in [27,29]. From an integrable systems viewpoint, the Somos-4 recurrence (2) arises as a 'bilinearization' of the following QRT map [45],

$$f_{n+1}f_nf_{n-1} = \alpha + \frac{\beta}{f_n}, \tag{3}$$

through the relation

$$f_n = \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2}, \tag{4}$$

which encodes the singular confinement of the QRT map [28,30], cf. [46] for 'multi-linear' forms of other integrable maps. Furthermore, Somos-4 is a special case of the much more general Gale-Robinson recurrence (9), which is a reduction from the Hirota–Miwa equation [19,22,40,57,58].

A deeper understanding of the Laurent property, for a wide class of recurrences, came with the work of Fomin and Zelevinski [14,15], and subsequent developments [1,18,37]. The algebraic combinatorial setting of cluster algebras has had profound impact in diverse areas of mathematics, such as algebraic Lie theory [20], Poisson geometry [21], higher Teichmüller theory [12], the representation theory of quivers and finite-dimensional algebras [5], and integrable systems [17], cf. the cluster algebra portal [13].

In this paper, which is an extended version of [24], we describe how one can obtain recurrences which possess the Laurent property, such as (2), from equations that are singularity confining, such as (3), which is different than via a transformation such as (4). Starting from a rational recurrence (integrable or not), we homogenize to get a polynomial map. Using an ultra-discretization we first determine the multiplicities of the divisors of its components under iteration. This is then used to derive recurrence relations for the sequences of divisors. Clearly, by definition, the divisors are polynomial. Hence one expects the derived recurrence to possess the Laurent property.

It is not a priori clear what type of recurrence relation one would get out of such a procedure. Two characteristic properties of discrete integrable systems are slow growth, and singularity confinement. Slow growth (= low complexity = vanishing algebraic entropy) is a better indicator of the integrability of a mapping [2–4,11,42,54] than singular confinement [26]. However, it is the latter property which allows us to say something about the kind of recurrences our method produces. For rational maps with singularity confinement, the reduced denominators depend on a fixed number of divisors as well as on the initial values.

The reason for this is that a single divisor (singularity) occurs only finitely many times. It implies that the order of the derived recurrence relation is fixed.

For non-confining rational maps the order of the derived recurrence relation may not be fixed. We have included a non-integrable example, which gives rise to a polynomial recurrence whose order and degree grow unboundedly. The Laurent property is obtained, but it is trivial.

Finally, for a rational map that possesses the Laurent property the fixed number of divisors in the reduced denominators will be 0. In such cases the method provides a validation of Laurentness for free. In the light of this one could say that Laurent recurrences are ultra-confining, in that they confine their singularities before they occur.

We remark here that Viallet, independently, has also found recurrence relations for sequences of divisors [55]. In particular, he obtains recurrences of fixed order from non-integrable confining maps [55, Sections 3.5, 3.6], and he presented a recurrence where the order grows unboundedly, obtained from a linearizable map [55, Section 3.8]. It is worth mentioning that in all cases considered in the present paper, as well as in the ones considered in [55], the coefficients (which depend on the initial values) turn out to be periodic functions. The reason for this is not yet understood.

In Section 2 we provide a brief account of the main method, which uses homogenization, ultra-discretization, and a technique that was introduced in [34], for which we coin the phrase *recursive factorization*. The method is explained in more detail by the examples in the subsequent sections. In Section 3 we show how the QRT-map (3), via recursive factorization, gives rise to a Somos-4 recurrence of the form (2) but where the coefficients are now functions of the initial values of the QRT-map, $\alpha = \alpha_n(f_1,f_2), \beta = \beta_n(f_1,f_2)$, which satisfy the periodicity conditions $\alpha_{n+p} = \alpha_n$, $\beta_{n+p} = \beta_n$ with $p = 8$. Similarly we show how another QRT-map yields a Somos-5 recurrence where the coefficients are periodic functions with period $p = 7$. In Section 4 we follow the same procedure starting with the Somos-4/5 sequences themselves. Surprisingly, or not, they give rise to Somos-4 and Somos-5 recurrences with more general periodic coefficients than those obtained in Section 3. Explicitly we have found

$$c_{n+2}c_{n-2} = \alpha_n c_{n+1}c_{n-1} + \beta_n c_n^2, \tag{5}$$

with coefficients

$$\alpha_n = \alpha \prod_{i=1}^{4} \tau_i^{p_{n-i}}, \quad \beta_n = \beta \prod_{i=1}^{4} \tau_i^{q_{n-i}}. \tag{6}$$

where[1] $p_{\bmod 8} = [1,0,0,1,0,0,1,0]$, $q_{\bmod 8} = [0,0,1,0,1,0,0,2]$, and

$$d_{n+3}d_{n-2} = \gamma_n d_{n+2}d_{n-1} + \delta_n d_n d_{n+1}, \tag{7}$$

with coefficients

$$\gamma_n = \gamma \prod_{i=1}^{5} \sigma_i^{r_{n-i}}, \quad \delta_n = \delta \prod_{i=1}^{5} \sigma_i^{s_{n-i}}. \tag{8}$$

where $r_{\bmod 7} = [1,0,0,0,1,0,0]$, $s_{\bmod 7} = [0,0,1,0,0,1,1]$. Both Equations (5) and (7) are special cases of a non-autonomous Gale-Robinson recurrence, cf. [19], with $v_1 + u_1 = v_2 + u_2 = w$,

$$h_n h_{n+w} = \alpha_n h_{n+v_1} h_{n+u_1} + \beta_n h_{n+v_2} h_{n+u_2}, \tag{9}$$

which is a reduction of the Hirota–Miwa equation [58]. Moreover, they satisfy the integrability condition,

$$\alpha_n \alpha_{n+w} \beta_{n+v_1} \beta_{n+u_1} = \alpha_{n+v_2} \alpha_{n+u_2} \beta_n \beta_{n+w}, \tag{10}$$

which is equivalent to Laurentness, see [40].

Integrable maps with periodic coefficients have appeared in the setting of QRT-type maps, and are described in the general context of pencils of biquadratic curves in [47], where there are references to other examples of non-QRT maps with periodic coefficients. It is worth noting that the condition (10) allows much more general behaviour than just periodic: it includes discrete Painlevé equations of q-type. Conditions like (10), and the associated discrete Painlevé equations, have recently been found to arise from the theory of cluster algebras and Y-systems [31,43]. Interestingly, the particular periods of the coefficients in (5) and (7) relate to the periods of the corresponding ultra-discrete QRT-maps (52) and (62), cf. [41].

In Section 5 we consider the first two members of the hierarchy of equations

$$\left(\sum_{k=0}^{N} u_{n+k}\right)\left(\prod_{l=1}^{N-1} u_{n+l}\right) = \phi. \tag{11}$$

which was introduced in [7], and whose degree growth has been studied in [23]. For $N = 2$ the map is another QRT-map, from which we obtain the fifth order Laurent recurrence

$$e_{n+5}e_{n+2}^2 e_{n+1} + e_{n+4}e_{n+3}^2 e_n + e_{n+4}^2 e_{n+1}^2 = \phi e_{n+3}^2 e_{n+2}^2. \tag{12}$$

For $N = 3$ we find that the ultra-discretization of the homogenized system does not yield a sharp bound on the multiplicities of the second divisor. Using primes as initial values enables us to iterate the system sufficiently many times to formulate a conjecture for these multiplicities. Via recursive factorization we then arrive at the following sixth order Laurent recurrence with periodic coefficients,

$$\frac{k_{n+2}}{k_{n-1}}(\epsilon_n k_{n-3}k_n^2 + \epsilon_{n+1}k_{n-2}^2 k_{n+1}) + \frac{k_{n-2}}{k_{n+1}}(\epsilon_{n+2}k_{n-1}k_{n+2}^2 + \epsilon_{n+3}k_n^2 k_{n+3}) = \frac{\phi k_n^3}{\epsilon_{n+1}\epsilon_{n+2}}$$

with $\epsilon_n = u_2^{\zeta_n}$ and $\zeta_{\bmod 8} = [0, 1, 0, -1, -1, 2, -1, -1]$.

In the last section we consider two distinct choices for the parameters in the generalized Lyness equation [6].

$$w_{n+3}w_n = \mu + \nu w_{n+1} + w_{n+2}. \tag{13}$$

The integrable subcase, $\nu = 1$, gives rise to the Laurent recurrence

$$z_{n+3}z_{n-2}z_{n-7} = \kappa_n z_{n-1}z_{n-2}z_{n-3} + \tau_n z_{n-1}z_{n+1}z_{n-6} + \sigma_n z_{n+2}z_{n-3}z_{n-5}, \tag{14}$$

where

$$\kappa_n = \mu \prod_{i=1}^{3} w_i^{\delta_{n-i}+\delta_{n-i+1}}, \quad \tau_n = \prod_{i=1}^{3} w_i^{\delta_{n-i+3}} \quad \text{and} \quad \sigma_n = \prod_{i=1}^{3} w_i^{\delta_{n-i+6}},$$

are periodic functions with $\delta_{\bmod 8} = [0,1,0,1,0,0,0,0]$. The non-integrable subcase, $\nu \neq 1$, gives rise to

$$z_n = \mu \left(\prod_{i=1}^{n-1} z_i^{\delta_{n-i-2}+\delta_{n-i-3}} \right) + \nu \left(\prod_{i=1}^{n-1} z_i^{\delta_{n-i}} \right) + \left(\prod_{i=1}^{n-1} z_i^{\delta_{n-i-5}} \right). \tag{15}$$

Here the Laurent property is trivially satisfied.

We stress that it is not surprising that the derived recurrences possess the Laurent property. We know a priori that they produce polynomials and there has to be a good reason for that to happen, cf. [8]. On the other hand, the Laurentness itself might not be enough to prove the polynomiality. The Laurent recurrence generates the divisors of the numerators and denominators of a rational map, which depend on both the parameters and the initial values of the integrable equation. For the periodic Somos sequences, and for the Lyness Laurent recurrence (14), this dependence is realized in the coefficients from the Laurent recurrence and we can start the recurrence with unit initial values. Thus, in this case, the polynomiality of the divisors is completely explained by the Laurentness of the recurrence. For the recurrences we have obtained from the DTKQ equations this is not the case. Here we have to initialize the recurrences with initial values that depend in a specific way on the initial values of the DTKQ equation. Therefore in these cases the Laurentness of the recurrences is not enough to explain the polynomiality of the divisors. One would need a strong Laurent property such as given for Somos-4/5 in [32]. This issue is left open for future research.

2. Ultra-discrete limits and recursive factorization

Given a rational recurrence (1) one can set $u_n = a_n/b_n$, and thus obtain a system of recurrences for a sequences of pairs of polynomials $\{a_n, b_n\}_{n=1}^{\infty}$. Such a system has two ultra-discrete versions: In the max-plus algebra one gets an upper bound on the growth of the degrees of the polynomials a_n and b_n, and the min-plus algebra yields a lower bound on the multiplicities of their divisors.

- The max-plus system is obtained by considering degrees. Let p, q, r be polynomials. The degree of $pq + r$ satisfies

$$\deg(pq + r) \leq \max\left(\deg(p) + \deg(q), \deg(r)\right).$$

- The min-plus system is obtained by considering the multiplicities of divisors. The multiplicity of any divisor f of $pq + r$ satisfies

$$\mathrm{mul}_f(pq + r) \geq \min\left(\mathrm{mul}_f(p) + \mathrm{mul}_f(q), \mathrm{mul}_f(r)\right).$$

For a discussion of ultra-discretization as a limiting procedure, see [53].

The degree of u_n is obtained from the degree of a_n (or b_n) minus the degree of the greatest common divisor $g_n = \gcd(a_n, b_n)$. Thus, one has to control the divisors of a_n and b_n. By iterating the system finitely many times and using the observed factorization as initial values in the ultra-discrete system for multiplicities, one obtains a lower bound on the multiplicities of divisors. In many cases this lower bound on the multiplicities is sharp. In any case, by recursively defining the next divisor to be the quotient of a term in the sequence after division by the previous divisors, one produces an exact factorization of the polynomial sequences (although not necessarily into irreducible factors). For example, if no new divisors appear in b_n we can write, in terms of a sequence of divisors $\{c_i\}_{i=1}^{\infty}$,

$$b_n = \sum_{i=1}^{n-1} c_i^{m_n^b(c_i)},$$

where $m_n^b(c_i)$ denotes the multiplicity of the ith divisor c_i in b_n. And, the nth divisor c_n is *defined* by

$$a_n = c_n \sum_{i=1}^{n-1} c_i^{m_n^a(c_i)}.$$

In other cases, new divisors do appear in b_n and the sequences a_n and b_n may be defined in terms of two sequences of divisors $\{c_i\}$ and $\{d_i\}$. If one is after degree growth one now writes the degree of a_n (or b_n) as a convolution of the degrees of the divisors and their multiplicities. Using (the solution to) the ultra-discrete degree recurrence one may then obtain a recurrence for the degrees of the divisors and, when all but finitely many divisors are common, retrieve an upper bound on the growth of degrees of u_n [23,34].

The idea of recursive factorization is, as far as we are aware, first published in the paper [34] where it was used to establish polynomial upper bounds on the growth of degrees of rational mappings. Although the max-plus ultra-discretization was used to bound the degrees of a_n's and b_n's, the multiplicities in the factorization were obtained from a recursion formula for the multiplicities of the divisors of the greatest common divisor $g_n = \gcd(a_n, b_n)$. This is not always sufficient. In [23] the min-plus ultra-discretization was used to find a lower bound on the multiplicities of the divisors, and so to obtain a factorization formula for the iterates of the Nth order DTKQ map (11). This was subsequently used to prove an upperbound on the growth of their degrees.

In this paper we obtain recurrence relations for the sequence of divisors by substituting the factorizations into the system of recurrences for the polynomial sequences $\{a_n\}$ and $\{b_n\}$. When all but a fixed number of divisors are common, this yields a nonlinear rational recurrence for the divisors $\{c_n\}$. As the divisors c_n are polynomial, we expect the recurrence to possess the Laurent property. If the number of divisors that are not common grows unboundedly (i.e. when the recurrence is not confining) the resulting recurrence does not have a fixed order, cf. Section 6. If one starts with a recurrence (1) that has the Laurent property, all divisors but powers of the initial variables, will be common to both a_n and b_n for all n. This then proves the Laurent property.

In a recent paper by Viallet [55], an alternative approach is taken. The maps are considered projectively and hence all common divisors are divided out. Viallet determines the form of the iterates, in terms of what he calls blocks, by iteration of the map until it stabilizes. He then poses algebraic relations for the blocks, i.e. recurrence relations for the divisors, and proves the validity of these relations and the stability of the form of the iterates simultaneously by induction. Another difference between the work of Viallet and the present paper is that for a given kth order rational map he homogenizes the corresponding first order k-dimensional system. The result is that his map lives in $\mathbb{P}^k$ whereas, if we would divide out common divisors, we would work in the kth Cartesian power of $\mathbb{P}^1$. A given divisor (block) will appear as a divisor of an earlier iterate in $\mathbb{P}^k$. Thus, when taking divisors along, their multiplicity grows faster than in $(\mathbb{P}^1)^k$ which is computationally a disadvantage. Other than that, this difference in homogenization is not a fundamental one. Both approaches yield the exact same recurrence relations.

We hope to further convince the reader of the usefulness of taking ultra-discretization limits in the study of growth of degrees and multiplicities of divisors with one more example. In [55, Section 3.9] Viallet mentions an 'unruly' model, $\phi : \mathbb{P}^3 \to \mathbb{P}^3$ given by the monomial map

$$[x, y, z, t] \mapsto [yt, zt, x^2, xt], \tag{16}$$

which he coins a limiting case for further developments. Monomial maps do not yield recurrence relations for its divisors because the only divisors that will appear are the ones that are already there, namely the initial values, i.e. the ith component of an iterate will have the form $x^{\delta_i^x} y^{\delta_i^y} z^{\delta_i^z} t^{\delta_i^t}$. However, one may study the sequences of degrees. As for monomial maps the degree sequences coincide with the sequences of multiplicities, the recursion relations for these sequences satisfy both the max-plus and the min-plus ultra-discretizations and so the $\leq$ and $\geq$ coincide in $=$. The degree sequences for the map (16) are given by the piecewise linear map in $\mathbb{N}^4$

$$\begin{pmatrix} \delta_1^s \\ \delta_2^s \\ \delta_3^s \\ \delta_4^s \end{pmatrix} \mapsto \begin{pmatrix} \delta_2^s + \delta_4^s \\ \delta_3^s + \delta_4^s \\ 2\delta_1^s \\ \delta_1^s + \delta_4^s \end{pmatrix} - \min\left(\delta_2^s + \delta_4^s, \delta_3^s + \delta_4^s, 2\delta_1^s, \delta_1^s + \delta_4^s\right) \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \tag{17}$$

where the second term on the right takes care of dividing out the common divisors. The initial values are $\delta^x = (1, 0, 0, 0)$, $\delta^y = (0, 1, 0, 0)$, $\delta^z = (0, 0, 1, 0)$, $\delta^t = (0, 0, 0, 1)$. We don't know how to obtain the algebraic entropy from a recurrence such as (17). For a description of the complexity of degree growth in monomial maps we refer the reader to [25, Proposition 7.3], where the map (16) was given as a counter-example to a conjecture by Bellon and Viallet that the degree sequence of any rational map satisfies a linear recurrence with constant coefficients.

3. From QRT maps to Somos-4/5 recurrences with periodic coefficients

In this section we show how by homogenization, an ultra-discrete limit and recursive factorization the QRT-map (3) leads to a special case of periodic Somos-4, Equation (5). A similar result for Somos-5 is also given.

3.1. To periodic Somos-4

We substitute $f_n = a_n/b_n$ in (3). This gives

$$\frac{a_{n+1}}{b_{n+1}} = \frac{w_{n+1} b_n b_{n-1}}{a_{n-1} a_n^2},$$

with $w_{n+1} := \alpha a_n + \beta b_n$, from which we obtain a system of recurrences for polynomial sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$,

$$a_{n+1} = w_{n+1} b_n b_{n-1}, \tag{18}$$
$$b_{n+1} = a_{n-1} a_n^2. \tag{19}$$

This we supplement with initial values $a_1 = f_1$, $a_2 = f_2$, $b_1 = b_2 = 1$. Iterating (18) and (19) three more times give us:

$$\begin{gathered}
a_{n+2} = a_{n-1} a_n^2 b_n r_1,\; b_{n+2} = a_n b_{n-1}^2 b_n^2 w_{n+1}^2,\\
a_{n+3} = a_{n-1} a_n^4 b_{n-1}^2 b_n^3 r_2 w_{n+1}^2,\; b_{n+3} = a_{n-1}^2 a_n^4 b_{n-1} b_n^3 w_{n+1} r_1^2,\\
a_{n+4} = a_{n-1}^3 a_n^9 b_{n-1}^4 b_n^8 r_2^2 r_3 w_{n+1}^4,\; b_{n+4} = a_{n-1}^3 a_n^{10} b_{n-1}^4 b_n^7 r_2^2 w_{n+1}^4,
\end{gathered}$$

where $\{r_i\}_{i=1}^{3}$, are irreducible polynomials in a_{n-1}, b_{n-1}, a_n, b_n, α and β. We observe the following factorization properties: w_{n+1} does not divide a_{n+2}, it divides b_{n+2} and a_{n+3} with multiplicity 2, it divides b_{n+3} with multiplicity 1, and w_{n+1} is a divisor of both a_{n+4} and b_{n+4} with multiplicity 4. Furthermore, from (18) and (19), we find the following ultra-discrete system of recurrences for multiplicities:

$$\begin{aligned}
m_{n+2}^a &\geq \min\left(m_{n+1}^a, m_{n+1}^b\right) + m_n^b + m_{n+1}^b,\\
m_{n+2}^b &= m_n^a + 2m_{n+1}^a.
\end{aligned}$$

where $m_i^p(f)$ denotes the multiplicity of a polynomial f in polynomial p_i and we have suppressed the dependence on f. Using the equal sign in the first equation, we get a lower bound for the multiplicities, which we denote using Euler's fraktur typesetting. Thus, we will employ the following system:

$$\begin{aligned}
\mathfrak{m}_{n+2}^a &= \min\left(\mathfrak{m}_{n+1}^a, \mathfrak{m}_{n+1}^b\right) + \mathfrak{m}_n^b + \mathfrak{m}_{n+1}^b,\\
\mathfrak{m}_{n+2}^b &= \mathfrak{m}_n^a + 2\mathfrak{m}_{n+1}^a.
\end{aligned} \tag{20}$$

To get a lower bound for the multiplicity of w_k ($k > 2$) in the sequences $\{a_n\}$ and $\{b_n\}$, we solve (20) with the following initial values: $\mathfrak{m}_{k+1}^a = 0$, $\mathfrak{m}_{k+1}^b = 2$, $\mathfrak{m}_{k+2}^a = 2$, $\mathfrak{m}_{k+2}^b = 1$ and $\mathfrak{m}_{k+3}^a = \mathfrak{m}_{k+3}^b = 4$. We find, for $n \geqslant k+3$, that $\mathfrak{m}_n^a(w_k) = \mathfrak{m}_n^b(w_k) = \mathfrak{m}_{n-k}$, where

$$\mathfrak{m}_1 = 0,\; \mathfrak{m}_2 = 2,\; \mathfrak{m}_{n+1} = 2\mathfrak{m}_n + \mathfrak{m}_{n-1}.$$

This can be seen by taking $\mathfrak{m}_k^a = \mathfrak{m}_k^b$ in the right hand sides of (20). One finds equality and hence $\mathfrak{m}_{n+2}^a = \mathfrak{m}_{n+2}^b$. We define sequences $\{\mathfrak{m}_n^a(c_i)\}_{n=1}^{\infty}$ and $\{\mathfrak{m}_n^b(c_i)\}_{n=1}^{\infty}$, for $i \in \{1, 2\}$, by (20) and initial values $\mathfrak{m}_j^a(c_i) = \delta_{ij}$ and $\mathfrak{m}_j^b(c_i) = 0$.

The polynomials a_n and b_n can be expressed in terms of a sequence $\{c_n\}_{n=1}^{\infty}$, of polynomials in a_1, a_2, α and β. Each polynomial c_n is defined as the quotient of a_n after division by powers of c_i for $i < n$ as follows,

$$a_n = \begin{cases} c_n & \text{if } n \leqslant 3,\\ c_1 c_2^2 c_4 & \text{if } n = 4,\\ c_1^{\mathfrak{m}_n^a(c_1)} c_2^{\mathfrak{m}_n^a(c_2)} \left(\prod_{i=3}^{n-3} c_i^{\mathfrak{m}_{n-i}}\right) c_{n-2}^2 c_n & \text{if } n > 4. \end{cases} \tag{21}$$

It is clear that c_n is polynomial because $c_i|w_i$ for all $i > 4$ and hence $\mathfrak{m}_n^a(c_i) \geqslant \mathfrak{m}_n^a(w_i)$. We know that $\prod_{i=1}^n c_i^{\mathfrak{m}_n^b(c_i)}|b_n$. Taking b_n to be given by

$$b_n = \begin{cases} 1 & \text{if } n \leqslant 2, \\ c_{n-2}c_{n-1}^2 & \text{if } n \in \{3,4\}, \\ c_1^{\mathfrak{m}_n^b(c_1)} c_2^{\mathfrak{m}_n^b(c_2)} \left(\prod_{i=3}^{n-3} c_i^{\mathfrak{m}_{n-i}}\right) c_{n-2}c_{n-1}^2, & \text{if } n > 4, \end{cases} \tag{22}$$

we verify Equation (19) is satisfied. Thus, defining $g_n = \gcd(a_n, b_n)$ to be the greatest common divisor of a_n and b_n, we get

$$g_n = \prod_{i=1}^n c_i^{\mathfrak{m}_n^g(c_i)} = c_1^{\mathfrak{m}_n^g(c_1)} c_2^{\mathfrak{m}_n^g(c_2)} \left(\prod_{i=3}^{n-3} c_i^{\mathfrak{m}_{n-i}}\right) c_{n-2}, \tag{23}$$

where $\mathfrak{m}_n^g(c_i) = \min(\mathfrak{m}_n^a(c_i), \mathfrak{m}_n^b(c_i))$. Note, from $\frac{b_n}{g_n} = c_1^{\mathfrak{m}_n^b(c_1)-\mathfrak{m}_n^g(c_1)} c_2^{\mathfrak{m}_n^b(c_2)-\mathfrak{m}_n^g(c_2)} c_{n-1}^2$, it can be seen that the map (3) does not posses the Laurent property, but that it does confine singularities. Indeed, the singularities from the previous iterate are still present but all others have disappeared.

Considering the lower bounds for the multiplicities of c_1, c_2 in a_n and b_n, we observe the following differences are periodic.

Lemma 1: *We have:*

$$\mathfrak{m}_k^a(c_i) - \mathfrak{m}_k^b(c_i) = \begin{cases} \nu_k & \text{if } i = 1, \\ \nu_{k-3} & \text{if } i = 2, \end{cases}$$

where $\nu_{\bmod 8} = [1, 0, -1, 1, -1, 0, 1, -2]$.

Proof: By induction. Suppose we have $\mathfrak{m}_k^a(c_1) = \mathfrak{m}_k^b(c_1) + \nu_k$ and $\mathfrak{m}_{k-1}^a(c_1) = \mathfrak{m}_{k-1}^b(c_1) + \nu_{k-1}$. Then

$$\begin{aligned} \mathfrak{m}_{k+1}^a(c_1) &= \min(\mathfrak{m}_k^a(c_1), \mathfrak{m}_k^b(c_1)) + \mathfrak{m}_{k-1}^b(c_1) + \mathfrak{m}_k^b(c_1) \\ &= 2\mathfrak{m}_k^b(c_1) + \mathfrak{m}_{k-1}^b(c_1) + \min(\nu_k, 0), \text{ and} \\ \mathfrak{m}_{k+1}^b(c_1) &= \mathfrak{m}_{k-1}^a(c_1) + 2\mathfrak{m}_k^a(c_1) \\ &= 2\mathfrak{m}_k^b(c_1) + \mathfrak{m}_{k-1}^b(c_1) + 2\nu_k + \nu_{k-1}. \end{aligned}$$

One verifies that $\min(\nu_k, 0) - 2\nu_k - \nu_{k-1} = \nu_{k+1}$. For c_2 the same equation is obtained (with $\nu_k \to \nu_{k-3}$). □

From (21), (22), (23), it follows that

$$\alpha_n := \frac{\alpha a_n}{c_n c_{n-2} g_n} \quad \text{and} \quad \beta_n := \frac{\beta b_n}{c_{n-1}^2 g_n} \tag{24}$$

are polynomials in c_1 and c_2. As a corollary to Lemma 1, it follows that α_n and β_n are periodic sequences of period 8, which is the period of the ultra-discrete QRT-map (52), cf. [41].

Corollary 2: *We have:*

$$\alpha_n = \alpha c_1^{p_n} c_2^{p_{n-3}} \quad \text{and} \quad \beta_n = \beta c_1^{q_n} c_2^{q_{n-3}}, \tag{25}$$

with $p_{\ mod\ 8} = [1, 0, 0, 1, 0, 0, 1, 0]$ *and* $q_{\ mod\ 8} = [0, 0, 1, 0, 1, 0, 0, 2]$.

Proof: We have:

$$\alpha_n = \frac{\alpha a_n}{g_n c_n c_{n-2}} = \alpha c_1^{\mathfrak{m}_n^a(c_1)-\mathfrak{m}_n^g(c_1)} c_2^{\mathfrak{m}_n^a(c_2)-\mathfrak{m}_n^g(c_2)},$$

where

$$\mathfrak{m}_n^a - \mathfrak{m}_n^g = \begin{cases} \mathfrak{m}_n^a - \mathfrak{m}_n^b & \text{if } \mathfrak{m}_n^a - \mathfrak{m}_n^b > 0, \\ 0 & \text{if } \mathfrak{m}_n^a - \mathfrak{m}_n^b \leqslant 0. \end{cases}$$

Therefore,

$$\mathfrak{m}_n^a(c_i) - \mathfrak{m}_n^g(c_i) = \begin{cases} p_n & \text{if } i = 1, \\ p_{n-3} & \text{if } i = 2, \end{cases}$$

where $p_k = \max(0, v_k)$. Similarly, we have:

$$\beta_n = \frac{\beta b_n}{g_n c_{n-1}^2} = \beta c_1^{\mathfrak{m}_n^b(c_1)-\mathfrak{m}_n^g(c_1)} c_2^{\mathfrak{m}_n^b(c_2)-\mathfrak{m}_n^g(c_2)},$$

where

$$\mathfrak{m}_n^b(c_i) - \mathfrak{m}_n^g(c_i) = \begin{cases} q_n & \text{if } i = 1, \\ q_{n-3} & \text{if } i = 2, \end{cases}$$

with $q_k = \max(0, -v_k)$. □

Theorem 3: *The polynomials c_n, as defined by (21), satisfy*

$$c_3 = \alpha c_2 + \beta, \quad c_4 = \alpha c_3 + \beta c_1 c_2^2, \quad c_5 = \alpha c_1 c_2 c_4 + \beta c_3^2, \quad c_6 = \alpha c_5 c_3 + \beta c_1 c_4^2, \tag{26}$$

and, for $n \geqslant 6$,

$$c_{n-3} c_{n+1} = \alpha_n c_n c_{n-2} + \beta_n c_{n-1}^2. \tag{27}$$

Proof: Using Equations (18) and (19), initial values and (21), we find:

$$c_3 = a_3 = (\alpha a_2 + \beta b_2) b_1 b_2 = (\alpha c_2 + \beta).$$

Furthermore,

$$c_4 = \frac{a_4}{c_1^{\mathfrak{m}_4^a(c_1)} c_2^{\mathfrak{m}_4^a(c_2)}} = \frac{(\alpha a_3 + \beta b_3) b_2 b_3}{c_1 c_2^2} = \alpha c_3 + \beta c_1 c_2^2,$$

as $b_3 = c_1 c_2^2$, $\mathfrak{m}_4^a(c_1) = 1$ and $\mathfrak{m}_4^a(c_2) = 2$. Similarly, the formulae for c_5 and c_6 are obtained. Solving Equations (24) for a_n and b_n and substituting in Equation (18), we find:

$$c_{n-3} c_{n+1} = Z_n (\alpha_n c_n c_{n-2} + \beta_n c_{n-1}^2),$$

with

$$Z_n = \frac{\beta_{n-1} \beta_n}{\beta^2} \frac{\alpha}{\alpha_{n+1}} \frac{g_{n-1} g_n^2}{g_{n+1}} c_{n-1} c_{n-2}^2 c_{n-3}.$$

Substituting in Equation (19) gives us:

$$g_{n+1} = \frac{\beta}{\beta_{n+1}} \frac{\alpha_n^2 \alpha_{n-1}}{\alpha^3} g_{n-1} g_n^2 c_{n-1} c_{n-2}^2 c_{n-3},$$

which we use to simplify

$$Z_n = \frac{\alpha^4 \beta_{n-1} \beta_n \beta_{n+1}}{\alpha_{n-1} \alpha_n^2 \alpha_{n+1} \beta^3} = 1,$$

as $q_{n-1} + q_n + q_{n+1} = p_{n-1} + 2p_n + p_{n+1}$. □

The fact that the sequence $\{c_n\}_{n=1}^{\infty}$, with special initial values given by (26) and generated by the rational recurrence (27), is a polynomial sequence is curious. First of all, it follows from the definition of c_n given by (21) which is based on factorization properties of the QRT map (3). But there is a second explanation. When we express the coefficients, cf. Corollary 2, in terms of the initial values of the QRT-map (3), $c_1 = a_1 = f_1$ and $c_2 = a_2 = f_2$, i.e.

$$\alpha_n^f = \begin{cases} \alpha f_1 & \text{if } n \equiv 1 \bmod 8, \\ \alpha f_2 & \text{if } n \equiv 2 \bmod 8, \\ \alpha f_1 f_2 & \text{if } n \equiv 4, 7 \bmod 8, \\ \alpha & \text{if } n \equiv 3, 5, 6, 8 \bmod 8, \end{cases} \qquad \beta_n^f = \begin{cases} \beta & \text{if } n \equiv 1, 2, 4, 7 \bmod 8, \\ \beta f_1 f_2^2 & \text{if } n \equiv 3 \bmod 8, \\ \beta f_1 & \text{if } n \equiv 5 \bmod 8, \\ \beta f_2 & \text{if } n \equiv 6 \bmod 8, \\ \beta f_1^2 f_2 & \text{if } n \equiv 8 \bmod 8, \end{cases} \tag{28}$$

and supplement the recurrence

$$c_{n-3} c_{n+1} = \alpha_n^f c_n c_{n-2} + \beta_n^f c_{n-1}^2. \tag{29}$$

with initial values $c_i = 1$ for $i \in \{-1, 0, 1, 2\}$ we find the following expressions

$$c_3 = \alpha f_2 + \beta, \quad c_4 = \alpha c_3 + \beta f_1 f_2^2, \quad c_5 = \alpha f_1 f_2 c_4 + \beta_3^2, \quad c_6 = \alpha c_5 c_3 + \beta f_1 c_4^2,$$

which agree with (26). Therefore, the fact that the sequence consist of polynomials is fully explained by the Laurent property of (29), cf. Section 4.3.

3.2. To periodic Somos-5

We will now show how the QRT-map

$$h_{n+1} h_n h_{n-1} = \gamma h_n + \delta, \tag{30}$$

which is related to Somos-5,

$$\sigma_{n+3} \sigma_{n-2} = \gamma \sigma_{n+2} \sigma_{n-1} + \delta \sigma_{n+1} \sigma_n, \tag{31}$$

via the transformation, cf. [29],

$$h_n = \frac{\sigma_{n+2} \sigma_{n-1}}{\sigma_{n+1} \sigma_n}, \tag{32}$$

leads to a special case of periodic Somos-5, Equation (7). Substituting $h_n = a_n/b_n$, the homogenized system for numerators and denominators is given by:

$$\begin{aligned} a_{n+1} &= v_{n+1} b_{n-1}, \\ b_{n+1} &= a_n a_{n-1}, \end{aligned} \tag{33}$$

where $v_{n+1} := \gamma a_n + \delta b_n$. We take $\{b_i = 1\}_{i=1}^2$, so that $\{a_i = h_i\}_{i=1}^2$. Iterating (33), we find:

$$\begin{aligned} a_{n+2} &= b_n s_1,\; b_{n+2} = v_{n+1} b_{n-1} a_n, \\ a_{n+3} &= a_n a_{n-1} s_2,\; b_{n+3} = b_n b_{n-1} s_1 v_{n+1}, \\ a_{n+4} &= a_n b_{n-1} s_3 v_{n+1},\; b_{n+4} = a_n a_{n-1} b_n s_1 s_2, \\ a_{n+5} &= a_n b_{n-1} b_n s_1 s_4 v_{n+1},\; b_{n+5} = a_{n-1} a_n^2 b_{n-1} s_2 s_3 v_{n+1}, \end{aligned}$$

where $\{s_i\}_{i=1}^4$ are irreducible polynomials in $\{a_{n+i}, b_{n+i}\}_{i=-1}^0$, δ, and γ. In addition, from (33), the ultra-discrete system of recurrences for a lower bound on the multiplicities is:

$$\begin{aligned} \mathfrak{m}_{n+1}^a &= \min(\mathfrak{m}_n^a, \mathfrak{m}_n^b) + \mathfrak{m}_{n-1}^b, \\ \mathfrak{m}_{n+1}^b &= \mathfrak{m}_n^a + \mathfrak{m}_{n-1}^a. \end{aligned} \tag{34}$$

To get a lower bound for the multiplicity of v_k $(k > 3)$ in the sequences $\{a_n\}$ and $\{b_n\}$, we solve (34) with initial values: $\mathfrak{m}_{k+1}^a = \mathfrak{m}_{k+2}^a = \mathfrak{m}_{k+3}^b = 0$ and $\mathfrak{m}_{k+1}^b = \mathfrak{m}_{k+2}^b = \mathfrak{m}_{k+3}^a = \mathfrak{m}_{k+4}^a = \mathfrak{m}_{k+4}^b = 1$. We find, for all $n \geqslant k+4$, that $\mathfrak{m}_n^a(v_k) = \mathfrak{m}_n^b(v_k) = \mathfrak{m}_{n-k-3}$ where

$$\mathfrak{m}_1 = 1,\; \mathfrak{m}_2 = 2,\; \mathfrak{m}_{n+2} = \mathfrak{m}_{n+1} + \mathfrak{m}_n.$$

For $i \in \{1, 2\}$ we define sequences $\mathfrak{m}_n^a(d_i)$ and $\mathfrak{m}_n^b(d_i)$ by (34) and the initial values $\mathfrak{m}_j^a(d_i) = \delta_{ij}$ and $\mathfrak{m}_j^b(d_i) = 0$. Then, a polynomial sequence $\{d_n\}_{n=1}^\infty$ is defined by

$$a_n = \begin{cases} d_n & \text{if } n \leqslant 4, \\ d_1 d_2 d_5 & \text{if } n = 5, \\ \left(\prod_{i=1}^2 d_i^{\mathfrak{m}_n^a(d_i)}\right)\left(\prod_{i=3}^{n-4} d_i^{\mathfrak{m}_{n-i-3}}\right) d_{n-3} d_n, & \text{if } n > 5, \end{cases} \tag{35}$$

and we have

$$b_n = \begin{cases} 1 & \text{if } n \leqslant 2, \\ d_{n-2} d_{n-1} & \text{if } n = 3, 4, \\ \left(\prod_{i=1}^2 d_i^{\mathfrak{m}_i^b(d_i)}\right)\left(\prod_{i=3}^{n-4} d_i^{\mathfrak{m}_{n-i-3}}\right) d_{n-2} d_{n-1} & \text{if } n \geqslant 5. \end{cases} \tag{36}$$

As in the previous section, the difference between the multiplicities of the initial divisors, d_1 and d_2, is periodic. We have:

$$\mathfrak{m}_k^a(d_i) - \mathfrak{m}_k^b(d_i) = \begin{cases} w_k & \text{if } i = 1, \\ w_{k-4} & \text{if } i = 2, \end{cases}$$

where $w_{\bmod 7} = [1, 0, -1, 0, 1, -1, -1]$, which can be proven by induction as was done in the proof of Lemma 1. From this, it follows that in terms of the initial values of the map (30), h_1 and h_2, we have

$$\gamma_n^h := \frac{\gamma a_n}{d_n d_{n-3} g_n} = \gamma h_1^{r_n} h_2^{r_{n-4}} \quad \text{and} \quad \delta_n^h := \frac{\delta b_n}{d_{n-1} d_{n-2} g_n} = \delta h_1^{s_n} h_2^{s_{n-4}}, \tag{37}$$

where $r_k = \max(0, w_k)$, $s_k = \max(0, -w_k)$, i.e.

$$r_{\bmod 7} = [1, 0, 0, 0, 1, 0, 0] \quad \text{and} \quad s_{\bmod 7} = [0, 0, 1, 0, 0, 1, 1]. \tag{38}$$

Solving (37) for a_n and b_n in terms of γ_n^h and δ_n^h and substituting into (33), we find the following recursion relations.

Theorem 4: *The sequence* $\{d_n\}_{n=1}^{\infty}$*, defined by* (35)*, satisfies*

$$\begin{aligned} &d_1 = h_1, \quad d_2 = h_2, \quad d_3 = \gamma h_2 + \delta, \quad d_4 = \gamma d_3 + \delta h_1 h_2, \\ &d_5 = \gamma d_4 + \delta h_2 d_3, \quad d_6 = \gamma h_1 h_2 d_5 + \delta d_3 d_4, \quad d_7 = \gamma d_3 d_6 + \delta h_1 d_4 d_5, \end{aligned} \tag{39}$$

and, for all $n \geqslant 8$,

$$d_{n-4} d_{n+1} = \gamma_n^h d_n d_{n-3} + \delta_n^h d_{n-1} d_{n-2}. \tag{40}$$

We note that (39) are obtained from (40) by taking initial values $d_i = 1$ for $i \in \{-2, -1, 0, 1, 2\}$. Therefore, the fact that $\{d_n\}_{n=1}^{\infty}$ is a sequence of polynomials is again explained by the Laurent property of (40), see Section 4.3.

Finally, we'd like to mention that the third order mapping [29, Equation 2.9],

$$u_{n+2} u_n^2 u_{n+1}^2 u_{n-1} = \gamma u_n u_{n+1} + \delta, \tag{41}$$

which is related to Somos-5 via

$$u_n = \frac{\sigma_{n+1} \sigma_{n-1}}{\sigma_n^2},$$

can be recursively factorized as $u_n = a_n / b_n$ with

$$a_n = \begin{cases} d_n & \text{if } n \leqslant 4, \\ d_1 d_2^2 d_3^3 d_5 & \text{if } n = 5, \\ d_1^{\mathfrak{m}_n^a(d_1)} d_2^{\mathfrak{m}_n^a(d_2)} d_3^{\mathfrak{m}_n^a(d_3)} \left(\prod_{i=4}^{n-3} d_i^{\mathfrak{m}_{n-i}}\right) d_{n-2}^3 d_n, & \text{if } n > 5, \end{cases} \tag{42}$$

and

$$b_n = \begin{cases} 1 & \text{if } n \leqslant 3, \\ d_{n-3} d_{n-2}^2 d_{n-1}^2 & \text{if } n = 4, 5, \\ d_1^{\mathfrak{m}_n^b(d_1)} d_2^{\mathfrak{m}_n^b(d_2)} d_3^{\mathfrak{m}_n^b(d_3)} \left(\prod_{i=4}^{n-3} d_i^{\mathfrak{m}_{n-i}}\right) d_{n-2}^2 d_{n-1}^2, & \text{if } n > 5, \end{cases} \tag{43}$$

where

$$\mathfrak{m}_1 = 0, \ \mathfrak{m}_2 = 3, \ \mathfrak{m}_3 = 7, \ \mathfrak{m}_{n+2} = 2\mathfrak{m}_{n+1} + 2\mathfrak{m}_n + \mathfrak{m}_{n-1},$$

and, for $i \in \{1,2,3\}$, $\{\mathfrak{m}_n^a(d_i)\}_{n=1}^\infty$ and $\{\mathfrak{m}_n^b(d_i)\}_{i=1}^\infty$ are defined by initial values $\{\mathfrak{m}_j^a(d_i) = \delta_{ij}, \mathfrak{m}_j^b(d_i) = 0\}_{i,j=1}^3$, and

$$\begin{aligned} \mathfrak{m}_{n+2}^a &= \min\left(\mathfrak{m}_n^a + \mathfrak{m}_{n+1}^a, \mathfrak{m}_n^b + \mathfrak{m}_{n+1}^b\right) + \mathfrak{m}_n^b + \mathfrak{m}_{n+1}^b + \mathfrak{m}_{n-1}^b, \\ \mathfrak{m}_{n+2}^b &= 2\mathfrak{m}_n^a + 2\mathfrak{m}_{n+1}^a + \mathfrak{m}_{n-1}^a. \end{aligned} \tag{44}$$

Here, the differences between the multiplicities of d_1, d_2, and d_3 are periodic sequences with period 14. We have:

$$\mathfrak{m}_k^a(d_i) - \mathfrak{m}_k^b(d_i) = \begin{cases} h_k & \text{if } i = 1, \\ h_k + h_{k+3 \bmod 14} & \text{if } i = 2, \\ h_{k-4} & \text{if } i = 3, \end{cases}$$

where $h_{\bmod 14} = [1,0,0,-1,1,0,-1,0,1,-1,0,0,1,-2]$, from which it follows that $\phi_n := \frac{a_n}{d_n d_{n-2} g_n}$ and $\psi_n := \frac{b_n}{d_{n-1}^2 g_n}$ are periodic with period 14. However, the coefficients of the periodic Somos-5 recurrence for the sequence $\{d_n\}_{n=1}^\infty$ defined by (42),

$$d_{n-3}d_{n+2} = \gamma_{n+2}^u d_{n-2}d_{n+1} + \delta_{n+2}^u d_n d_{n-1}, \tag{45}$$

turn out to have period 7,

$$\begin{aligned} \gamma_{n+2}^u &= \gamma \frac{\psi_{n-1}\psi_n\psi_{n+1}\psi_{n+2}}{\phi_{n-1}\phi_n\phi_{n+1}\phi_{n+2}} = \gamma u_1^{r_n} u_2^{r_n + r_{n-3}} u_3^{r_{n-3}}, \\ \delta_{n+2}^u &= \delta \frac{\psi_{n-1}\psi_n^2\psi_{n+1}^2\psi_{n+2}}{\phi_{n-1}\phi_n^2\phi_{n+1}^2\phi_{n+2}} = \delta u_1^{s_n} u_2^{s_n + s_{n-3}} u_3^{s_{n-3}}, \end{aligned}$$

with r, s as before. Thus, Equation (45) sits inside the periodic Somos-5 family mentioned in the introduction, Equation (7).

4. From Somos-4/5 recurrences to Somos-4/5 recurrences with (more general) periodic coefficients

In this section we apply our method to the Somos-4/5 sequences. We obtain the Somos sequences with periodic coefficients mentioned in the introduction, which are slightly more general than the ones obtained from QRT-maps in the previous section. Whereas in the previous section the differences between the multiplicities $\mathfrak{m}_n^a - \mathfrak{m}_n^b$ of the initial divisors were periodic functions of n. Here they satisfy ultra-discrete Somos-4/5 recurrences, which are not periodic.

4.1. Periodic Somos-4

By taking $\tau_n = a_n/b_n$ in Somos-4 we find the system of recurrences for polynomial sequences $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$:

$$a_{n+2} = w_{n+2} b_{n-2}, \tag{46}$$

$$b_{n+2} = b_{n+1} b_n^2 b_{n-1} a_{n-2}, \tag{47}$$

with $w_{n+2} := \alpha a_{n+1} b_n^2 a_{n-1} + \beta b_{n+1} a_n^2 b_{n-1}$. Taking $\{b_i = 1\}_{i=1}^4$, we have $\{a_i = \tau_i\}_{i=1}^4$. From (46) and (47), we get the following ultra-discrete system of recurrences for a lower bound on multiplicities:

$$\begin{aligned} \mathfrak{m}_{n+2}^a &= \min\left(\mathfrak{m}_{n+1}^a + 2\mathfrak{m}_n^b + \mathfrak{m}_{n-1}^a, \mathfrak{m}_{n+1}^b + 2\mathfrak{m}_n^a + \mathfrak{m}_{n-1}^b\right) + \mathfrak{m}_{n-2}^b, \\ \mathfrak{m}_{n+2}^b &= \mathfrak{m}_{n+1}^b + 2\mathfrak{m}_n^b + \mathfrak{m}_{n-1}^b + \mathfrak{m}_{n-2}^a. \end{aligned} \tag{48}$$

Iterating the recurrences (46), (47) four times gives us

$$\begin{aligned} a_{n+3} &= b_{n-1} b_{n+1} p_1,\ b_{n+3} = a_{n-2} a_{n-1} b_{n-1} b_n^3 b_{n+1}^3, \\ a_{n+4} &= a_{n-2} b_{n-1} b_n^4 b_{n+1}^3 p_2,\ b_{n+4} = a_{n-2}^3 a_{n-1} a_n b_{n-1}^3 b_n^7 b_{n+1}^6, \\ a_{n+5} &= a_{n-2}^3 a_{n-1} b_{n-1}^3 b_n^9 b_{n+1}^{10} p_3,\ b_{n+5} = a_{n-2}^6 a_{n-1}^3 a_n a_{n+1} b_{n-1}^6 b_n^{15} b_{n+1}^{13}, \\ a_{n+6} &= a_{n-2}^{10} a_{n-1}^3 a_n b_{n-1}^{10} b_n^{25} b_{n+1}^{23} w_{n+2} p_4,\ b_{n+6} = a_{n-2}^{13} a_{n-1}^6 a_n^3 a_{n+1} b_{n-2} b_{n-1}^{13} b_n^{32} b_{n+1}^{28} w_{n+2}, \end{aligned}$$

where p_1, p_2, p_3, p_4 are irreducible polynomials in $\{a_{n+i}, b_{n+i}\}_{i=-2}^1$, α and β. We obtain a lower bound for the multiplicity of w_k ($k > 4$) in the sequences $\{a_n\}$ and $\{b_n\}$, by solving (48) with the following initial values: $\mathfrak{m}_{k+i}^a = \mathfrak{m}_{k+i}^b = 0$, where $i \in \{1, 2, 3\}$ and $\mathfrak{m}_{k+4}^a = \mathfrak{m}_{k+4}^b = 1$. We find, for $n \geqslant k+1$,

$$\mathfrak{m}_n^a(w_k) = \mathfrak{m}_n^b(w_k) = \mathfrak{m}_{n-k},$$

where the integer sequence $\{\mathfrak{m}_n\}_{n=1}^\infty$ is defined by

$$\mathfrak{m}_{n+2} = \mathfrak{m}_{n+1} + 2\mathfrak{m}_n + \mathfrak{m}_{n-1} + \mathfrak{m}_{n-2},$$

and $\mathfrak{m}_1 = \mathfrak{m}_2 = \mathfrak{m}_3 = \mathfrak{m}_4 - 1 = 0$. We define sequences $\{\mathfrak{m}_n^a(c_i)\}_{n=1}^\infty$ and $\{\mathfrak{m}_n^b(c_i)\}_{n=1}^\infty$, for $i \in \{1, 2, 3, 4\}$, by (48) and the initial values $\{\mathfrak{m}_j^a(c_i) = \delta_{ij}, \mathfrak{m}_j^b(c_i) = 0\}_{i,j=1}^4$. Next, polynomials c_n are defined as a quotient of a_n, as follows,

$$a_n = \begin{cases} c_n & \text{if } n \leqslant 6, \\ \left(\prod_{i=1}^4 c_i^{\mathfrak{m}_n^a(c_i)}\right)\left(\prod_{i=5}^{n-1} c_i^{\mathfrak{m}_{n-i}}\right) c_n, & \text{if } n > 6, \end{cases} \tag{49}$$

and b_n can be expressed as

$$b_n = \begin{cases} 1 & \text{if } n \leqslant 4, \\ \left(\prod_{i=1}^4 c_i^{\mathfrak{m}_n^b(c_i)}\right) \prod_{i=5}^{n-1} c_i^{\mathfrak{m}_{n-i}}, & \text{if } n > 4. \end{cases}$$

Note that, with $g_n = \gcd(a_n, b_n)$ and $\mathfrak{m}_n^g(c_i) = \min\left(\mathfrak{m}_n^a(c_i), \mathfrak{m}_n^b(c_i)\right)$, we have

$$g_n = \left(\prod_{i=1}^4 c_i^{\mathfrak{m}_n^g(c_i)}\right) \prod_{i=5}^{n-1} c_i^{\mathfrak{m}_{n-i}} \text{ and } \frac{b_n}{g_n} = \prod_{i=1}^4 c_i^{\mathfrak{m}_n^b(c_i) - \mathfrak{m}_n^g(c_i)}, \tag{50}$$

which confirms that Somos-4 possesses the Laurent property. We next study the multiplicities of the divisors $\{c_i\}_{i=1}^4$. But first, let us define the ultra-discrete Somos-4 recurrence

$$r_{n+4} = -r_n + \max(r_{n+3} + r_{n+1}, 2r_{n+2}), \tag{51}$$

for which we take initial values $r_1 = -1$, $r_2 = r_3 = r_4 = 0$, cf. [17, Example 3.6]. The second difference of this sequence, $x_k = r_{k+2} - 2r_{k+1} + r_k$, is a periodic sequence of order 8. This follows by iteration of

$$x_{k+2} + 2x_{k+1} + x_k = \max(x_{k+1}, 0), \tag{52}$$

which itself is an ultra-discrete version of the QRT-map (3). We have

$$x_{\bmod 8} = [-1, 0, 1, -1, 1, 0, -1, 2].$$

Note, for general ultra-discrete QRT maps (as piecewise linear maps in $\mathbb{R}^2$, not just with integer values for dependent variables) Nobe proves the periodicity of all orbits, and gives explicit formulae for the periods in all cases [41].

Lemma 5: *For all $1 \leqslant i \leqslant 4$, we have: $\mathfrak{m}_n^b(c_i) - \mathfrak{m}_n^a(c_i) = r_{n-i+1}$.*

Proof: It is enough to prove the lemma for $i = 1$, because $\mathfrak{m}_n^a(c_i) = \mathfrak{m}_{n-1}^a(c_{i-1})$ and $\mathfrak{m}_n^b(c_i) = \mathfrak{m}_{n-1}^b(c_{i-1})$ for $i = \{2, 3, 4\}$ and $n > 1$. For brevity we omit the dependence of $\mathfrak{m}_k^a$ and $\mathfrak{m}_k^b$ on c_1. From initial values, we see that $\mathfrak{m}_n^b - \mathfrak{m}_n^a = r_n$ for $1 \leqslant n \leqslant 4$. According to the induction hypothesis, we may replace $\mathfrak{m}_l^a = \mathfrak{m}_l^b - r_l$, for $l \leqslant k$, in the right hand sides of (48), with $n = k - 1$. This gives

$$\begin{aligned}
\mathfrak{m}_{k+1}^a &= \min(\mathfrak{m}_k^b + 2\mathfrak{m}_{k-1}^b + \mathfrak{m}_{k-2}^b - r_k - r_{k-2}, \mathfrak{m}_k^b + 2\mathfrak{m}_{k-1}^b + \mathfrak{m}_{k-2}^b - 2r_{k-1}) + \mathfrak{m}_{k-3}^b \\
&= \mathfrak{m}_k^b + 2\mathfrak{m}_{k-1}^b + \mathfrak{m}_{k-2}^b + \mathfrak{m}_{k-3}^b + \min(-(r_k + r_{k-2}), -2r_{k-1}) \\
&= \mathfrak{m}_k^b + 2\mathfrak{m}_{k-1}^b + \mathfrak{m}_{k-2}^b + \mathfrak{m}_{k-3}^b - \max(r_k + r_{k-2}, 2r_{k-1}) \\
\mathfrak{m}_{k+1}^b &= \mathfrak{m}_k^b + 2\mathfrak{m}_{k-1}^b + \mathfrak{m}_{k-2}^b + \mathfrak{m}_{k-3}^b - r_{k-3}.
\end{aligned}$$

Thus, $\mathfrak{m}_{k+1}^b - \mathfrak{m}_{k+1}^a = -r_{k-3} + \max(r_k + r_{k-2}, 2r_{k-1}) = r_{k+1}$. □

Theorem 6: *For all $n > 4$, the polynomials c_i defined by (49) satisfy the Somos-4 recurrence*

$$c_{n-2}c_{n+2} = \alpha_n^\tau c_{n+1}c_{n-1} + \beta_n^\tau c_n^2, \tag{53}$$

with initial values $\{c_i = 1\}_{i=1}^4$ and periodic coefficients

$$\alpha_n^\tau = \alpha \prod_{i=1}^{4} \tau_i^{p_{n-i \bmod 8}}, \quad \beta_n^\tau = \beta \prod_{i=1}^{4} \tau_i^{q_{n-i \bmod 8}}, \tag{54}$$

where p and q are given in Corollary 2.

Proof: From (49), (50) and the initial values we obtain

$$c_5 = \alpha c_2 c_4 - \beta c_3, \quad c_6 = \alpha c_3 c_5 + \beta c_1 c_4, \quad c_7 = \alpha c_1 c_4 c_6 + \beta c_2 c_5, \quad c_8 = \alpha c_1 c_4 c_6 + \beta c_2 c_5. \tag{55}$$

Using Lemma 5, we find $a_n = g_n c_n$ and $b_n = (\prod_{i=1}^{4} c_i^{r_{n-i+1}})g_n$. Substituting these expressions in (46) gives, for $n > 8$

$$c_{n+2}g_{n+2} = g_{n-1}g_{n-2}g_n^2 g_{n+1}\left(\alpha c_{n+1}c_{n-1}\prod_{i=1}^{4} c_i^{2r_{n-i+1}+r_{n-i-1}} + \beta c_n^2 \prod_{i=1}^{4} c_i^{r_{n-i+2}+r_{n-i-1}+r_{n-i}}\right).$$

From (47), we find:

$$\prod_{i=1}^{4} c_i^{r_{n-i+3}} g_{n+2} = g_{n-1}g_{n-2}g_n^2 g_{n+1}\left(\prod_{i=1}^{4} c_i^{r_{n-i+2}+2r_{n-i+1}+r_{n-i}}\right) c_{n-2}.$$

Eliminating g_{n+2} from the above yields

$$c_{n+2}c_{n-2} = \alpha\left(\prod_{i=1}^{4} c_i^{r_{n-i+3}-r_{n-i+2}-r_{n-i}+r_{n-i-1}}\right) c_{n+1}c_{n-1} + \beta\left(\prod_{i=1}^{4} c_i^{r_{n-i+3}-2r_{n-i+1}+r_{n-i-1}}\right) c_n^2,$$

which can be expressed in terms of p and q, as follows,

$$r_{n-i+3} - r_{n-i+2} - r_{n-i} + r_{n-i-1} = x_{n-i+1} + x_{n-i} + x_{n-i-1} = p_{n-i \bmod 8},$$

and

$$r_{n-i+3} - 2r_{n-i+1} + r_{n-i-1} = x_{n-i+1} + 2x_{n-i} + x_{n-i-1} = q_{n-i \bmod 8}.$$

Taking unit initial values $\{c_i = 1\}_{i=1}^{4}$, the relations (55) are generated by (53). □

4.2. Periodic Somos-5

For Somos-5 we follow the same steps. Homogenising $\sigma_n = a_n/b_n$ gives

$$a_{n+3} = v_{n+3}b_{n-2}, \tag{56}$$
$$b_{n+3} = b_{n+2}b_{n-1}b_nb_{n+1}a_{n-2}, \tag{57}$$

where $v_{n+3} := \gamma a_{n+2}a_{n-1}b_nb_{n+1} + \delta a_na_{n+1}b_{n+2}b_{n-1}$. We take $\{b_i = 1\}_{i=1}^{5}$ and so $\{a_i = \sigma_i\}_{i=1}^{5}$. Iterating (56) and (57) five more times, we find:

$$\begin{aligned}
&a_{n+4} = b_{n+2}b_{n+1}b_{n-1}q_1,\ b_{n+4} = a_{n-2}a_{n-1}b_{n-1}b_n^2b_{n+1}^2b_{n+2}^2,\\
&a_{n+5} = a_{n-2}b_{n-1}b_n^2b_{n+1}^2b_{n+2}^2q_2,\ b_{n+5} = a_{n-2}^2a_{n-1}a_nb_{n-1}^2b_n^3b_{n+1}^4b_{n+2}^4,\\
&a_{n+6} = a_{n-2}^2a_{n-1}b_{n-1}^3b_n^3b_{n+1}^6b_{n+2}^5q_3,\ b_{n+6} = a_{n-2}^4a_{n-1}^2a_na_{n+1}b_{n-1}^4b_n^6b_{n+1}^7b_{n+2}^8,\\
&a_{n+7} = a_{n-2}^5a_{n-1}^2a_nb_{n-1}^6b_n^8b_{n+1}^{11}b_{n+2}^{12}q_4,\ b_{n+7} = a_{n-2}^8a_{n-1}^4a_n^2a_{n+1}a_{n+2}b_n^{12}b_{n+1}^{14}b_{n+2}^{15},\\
&a_{n+8} = a_{n-2}^{12}a_{n-1}^5a_n^2a_{n+1}b_{n-1}^{14}b_n^{18}b_{n+1}^{24}b_{n+2}^{25}v_{n+3}q_5,\\
&b_{n+8} = a_{n-2}^{15}a_{n-1}^8a_n^4a_{n+1}^2a_{n+2}b_{n-2}b_{n-1}^{15}b_n^{23}b_{n+1}^{27}b_{n+2}^{29}v_{n+3},
\end{aligned}$$

where $\{q_i\}_{i=1}^5$ are irreducible polynomials in $\{a_{n+i}, b_{n+i}\}_{i=-2}^2$, δ and γ. From (56) and (57), the system that gives a lower bound for multiplicities is:

$$\mathfrak{m}_{n+3}^a = \min\left(\mathfrak{m}_{n+2}^a + \mathfrak{m}_{n-1}^a + \mathfrak{m}_n^b + \mathfrak{m}_{n+1}^b, \mathfrak{m}_n^a + \mathfrak{m}_{n+1}^a + \mathfrak{m}_{n+2}^b + \mathfrak{m}_{n-1}^b\right) + \mathfrak{m}_{n-2}^b,$$
$$\mathfrak{m}_{n+3}^b = \mathfrak{m}_{n+2}^b + \mathfrak{m}_{n-1}^b + \mathfrak{m}_n^b + \mathfrak{m}_{n+1}^b + \mathfrak{m}_{n-2}^a. \tag{58}$$

To obtain a lower bound for $\mathfrak{m}_n^a(v_k)$ and $\mathfrak{m}_n^b(v_k)$, we solve (58) with the following initial values: $\mathfrak{m}_{k+i}^a = \mathfrak{m}_{k+i}^b = 0$ for all $i \in \{1, 2, 3, 4\}$ and $\mathfrak{m}_{k+5}^a = \mathfrak{m}_{k+5}^b = 1$. We find, for all $n \geqslant k + 1$,

$$\mathfrak{m}_n^a(v_k) = \mathfrak{m}_n^b(v_k) = \mathfrak{m}_{n-k},$$

where $\mathfrak{m}_{n+4} = \mathfrak{m}_{n+3} + \mathfrak{m}_{n+2} + \mathfrak{m}_{n+1} + \mathfrak{m}_n + \mathfrak{m}_{n-1}$ and $\mathfrak{m}_1 = \mathfrak{m}_2 = \mathfrak{m}_3 = \mathfrak{m}_4 = \mathfrak{m}_5 - 1 = 0$. Then, the formulae for a_n and b_n in terms of a new sequence $\{d_k\}_{k=1}^\infty$ are given as follows, with $n > 5$,

$$a_n = \left(\prod_{i=1}^5 d_i^{\mathfrak{m}_n^a(d_i)}\right)\left(\prod_{i=6}^{n-1} d_i^{\mathfrak{m}_{n-i}}\right) d_n, \quad b_n = \left(\prod_{i=1}^5 d_i^{\mathfrak{m}_n^b(d_i)}\right)\prod_{i=6}^{n-1} d_i^{\mathfrak{m}_{n-i}}, \tag{59}$$

where sequences $\{\mathfrak{m}_n^a(d_{i\leq 5})\}_{n=1}^\infty$ and $\{\mathfrak{m}_n^b(d_{i\leq 5})\}_{n=1}^\infty$ are defined by (58) and the initial values $\{\mathfrak{m}_j^a(d_i) = \delta_{ij}, \mathfrak{m}_j^b(d_i) = 0\}_{i,j=1}^5$. These formulae illustrate that Somos-5 possesses the Laurent property. The differences between the multiplicities of $c_{i\leq 5}$ can be expressed in terms of the ultra-discrete Somos-5 sequence defined by

$$t_{n+5} = -t_n + \max\left(t_{n+4} + t_{n+1}, t_{n+3} + t_{n+2}\right), \tag{60}$$

and initial values $t_1 = -1$, $\{t_i = 0\}_{i=2}^5$. The quantity

$$y_k = t_{k+3} - t_{k+2} - t_{k+1} + t_k, \tag{61}$$

which relates to (32), satisfies the ultra-discrete QRT-map, related to Equation (30),

$$y_{k+2} + y_{k+1} + y_k = \max\left(y_{k+1}, 0\right), \tag{62}$$

and is periodic of order 7, in fact we have $y_{\bmod 7} = [-1, 0, 1, 0, -1, 1, 1]$, see [16, Example 3.1] and the more general result in [41]. It follows from (58) that

$$\mathfrak{m}_n^b(d_i) - \mathfrak{m}_n^a(d_i) = t_{n-i+1}, \tag{63}$$

for all $1 \leqslant i \leqslant 5$.

Theorem 7: *Let r, s be given as in (38). For all $n > 7$, the polynomials $d_{n>5}$ defined by (59), satisfy the Somos-5 recurrence with periodic coefficients*

$$d_{n+3}d_{n-2} = \gamma_n d_{n+2}d_{n-1} + \delta_n d_n d_{n+1}, \tag{64}$$

where

$$\gamma_n = \gamma \prod_{i=1}^{5} \sigma_i^{r_{n-i \bmod 7}}, \quad \delta_n = \delta \prod_{i=1}^{5} \sigma_i^{s_{n-i \bmod 7}}, \tag{65}$$

and initial values $\{d_i = 1\}_{i=1}^{5}$.

Proof: Using (56), (57), initial values and (59), we find

$$\begin{aligned} &d_6 = \alpha\sigma_5\sigma_2 + \beta\sigma_3\sigma_4, \quad d_7 = \alpha d_6\sigma_3 + \beta\sigma_4\sigma_5\sigma_1, \quad d_8 = \alpha d_7\sigma_4 + \beta\sigma_2 d_6\sigma_5, \\ &d_9 = \alpha d_8\sigma_5\sigma_1 + \beta\sigma_3 d_7 d_6, \quad \text{and } d_{10} = \alpha d_9 d_6\sigma_2 + \beta\sigma_1\sigma_4 d_8 d_7. \end{aligned} \tag{66}$$

For all $n > 10$, from (59) and (63), we find $a_n = g_n d_n$ and $b_n = (\prod_{i=1}^{5} d_i^{t_{n-i+1}})g_n$. Substitution in Equation (56) gives

$$\frac{d_{n+3}g_{n+3}}{g_{n-2}g_{n-1}g_n g_{n+1}g_{n+2}} = \gamma\, d_{n+2}d_{n-1} \prod_{i=1}^{5} d_i^{t_{n-i+1}+t_{n-i+2}+t_{n-i-1}} + \delta d_n d_{n+1} \prod_{i=1}^{5} d_i^{t_{n-i+3}+t_{n-i}+t_{n-i-1}}.$$

From (57), we find:

$$\frac{g_{n+3}}{g_{n-2}g_{n-1}g_n g_{n+1}g_{n+2}} \prod_{i=1}^{5} d_i^{t_{n-i+4}} = \left(\prod_{i=1}^{5} d_i^{t_{n-i+3}+t_{n-i}+t_{n-i+1}+t_{n-i+2}}\right) d_{n-2}.$$

Eliminating g_n from the above equations yields the required result, as

$$\begin{aligned} t_{n-i+4} - t_{n-i+3} - t_{n-i} + t_{n-i-1} &= y_{n-i+1} + y_{n-i-1} = r_{n-i \bmod 7}, \\ t_{n-i+4} - t_{n-i+1} - t_{n-i+2} + t_{n-i-1} &= y_{n-i+1} + y_{n-i} + y_{n-i-1} = s_{n-i \bmod 7}. \end{aligned}$$

Moreover, taking $\{d_i = 1\}_1^5$, then (66) are generated by (64). □

4.3. On the Laurent property of periodic Somos-4/5 sequences

As the periodic Somos-4/5 sequences (53), (64) are special cases of Equation (9) and condition (10) is satisfied, they possess the Laurent property.

If we would not have had the Gale-Robinson equation at hand, or one wants a direct proof, this can be done. Actually, most of the work has already been done. Considering (53), the substitution $c_n = a_n/b_n$ yields the same system of recurrences (46), (47) for polynomials a_n and b_n. The only difference is that in the expression for w_{n+2}, α and β are now periodic functions of n with period 8. This means that the iteration of the recurrences (four more times) has to be repeated for different values of $n \equiv i \bmod 8$, with $i \in \{0, 1, 2, \ldots, 7\}$. For each value of i we found that w_{n+2} does not divide a_{n+k} or b_{n+k}, with $k = 3, 4, 5$, and that it does divide both a_{n+6} and b_{n+6}. As the system of recurrences is similar, the derived ultra-discrete system (48) is the same, polynomials c_n are defined by Equation (49), and the proof carries over. Also no surprises were found when iterating the system (56), (57) five more times, for $p = 7$ different values for $n \bmod p$.

5. From DTKQ equations to Laurent recurrences

The aim of this section is to show how the second and third order DTKQ equations give rise to recurrences that possess the Laurent property. The Nth order DTKQ equation,

$$\sum_{s=0}^{N} u_{n+s} \prod_{q=1}^{N-1} u_{n+q} = \phi, \tag{67}$$

was derived in [7], by applying the principle of duality for difference equations. It was shown to admit sufficiently many integrals to be completely integrable. The growth of the equations has been studied in [23].

5.1. From the second order DTKQ equation to a fifth order Laurent recurrence with four terms

In the case $N = 2$, the DTKQ equation is

$$u_{n+2} = \frac{\phi}{u_{n+1}} - u_n - u_{n+1}. \tag{68}$$

This is another example of a symmetric QRT map. The period of its ultra-discretization can be found in [41]. However, here the resulting Laurent system does not have periodic coefficients. In fact, that is the case for all additive QRT-maps [35].

Substituting $u_n = a_n/b_n$ in (68) and identifying the numerators and denominators, we get a system of recurrences for polynomial sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$:

$$a_{n+2} = \phi b_n b_{n+1}^2 - a_n a_{n+1} b_{n+1} - b_n a_{n+1}^2, \tag{69}$$

$$b_{n+2} = a_{n+1} b_n b_{n+1}, \tag{70}$$

with $a_1 = u_1$, $a_2 = u_2$, $b_1 = b_2 = 1$. Via ultra-discretization and recursive factorization, they are written in terms of a polynomial sequence $\{e_n\}$ as

$$a_n = \begin{cases} e_n & \text{if } n \leqslant 3, \\ e_n e_{n-3} \prod_{i=2}^{n-3} e_i^{\mathfrak{m}_{n-i-2}} & \text{if } n > 3, \end{cases}$$

$$b_n = \begin{cases} 1 & \text{if } n \leqslant 2, \\ e_2 & \text{if } n = 3, \\ e_{n-1} e_{n-2} \prod_{i=2}^{n-3} e_i^{\mathfrak{m}_{n-i-2}} & \text{if } n > 3, \end{cases} \tag{71}$$

with $\mathfrak{m}_1 = 2$, $\mathfrak{m}_2 = 6$, and $\mathfrak{m}_l = 2\mathfrak{m}_{l-1} + \mathfrak{m}_{l-2}$. More details can be found in [23] where a polynomial upperbound on the growth of (68) was obtained.

Theorem 8: *The polynomials e_n satisfy, for $n > 3$,*

$$\frac{e_{n-1}e_{n-5}}{e_{n-3}^2} + \frac{e_{n-1}^2 e_{n-4}^2}{e_{n-3}^2 e_{n-2}^2} + \frac{e_{n-4}e_n}{e_{n-2}^2} = \phi, \tag{72}$$

where

$$\{e_i = 1\}_{i=-1}^{1}, \quad e_2 = u_2 \quad \textit{and} \quad e_3 = \phi - u_1 u_2 - u_2^2. \tag{73}$$

Proof: From (69)–(70) and initial values, we obtain

$$e_2 = a_2, \quad e_3 = a_3 = \phi b_1 {b_2}^2 - a_1 a_2 b_2 - b_1 {a_2}^2 = \phi - u_1 u_2 - u_2^2.$$

Similarly, we find

$$e_4 = \phi u_2^2 - e_3 u_2^2 - e_3^2, \quad e_5 = \frac{\phi u_2^2 e_3^2 - e_4 e_3^2 - e_4^2}{u_2^2}, \quad e_6 = \frac{a_6}{e_3 g_6} = \frac{\phi e_3^2 e_4^2 - e_5 e_4^2 - e_5^2 u_2^2}{u_2 e_3^2}. \tag{74}$$

Now consider, for $n > 4$, replacing a_{n+i} by $e_{n+i} c_{n-3+i} g_{n+i}$ and b_{n+i} by $e_{n-1+i} e_{n-2+i} g_{n+i}$ in the right hand side of Equation (69):

$$e_{n+2} e_{n-1} g_{n+2} = g_n g_{n+1}^2 e_{n-1} e_{n-2} (\phi e_n^2 e_{n-1}^2 - e_n^2 e_{n-3} e_{n+1} - e_{n+1}^2 e_{n-2}^2).$$

From Equation (70) we find $g_n g_{n+1}^2 = \dfrac{g_{n+2}}{e_{n-1}^2 e_{n-2}^2}$ and these combine to give the recurrence equation for $e's$, (72). By taking $e_{-1} = e_0 = e_1 = 1$, the recurrence Equation (72) generates the above expressions (74). □

We could now recursively factorize the Equation (72), but if one just wants to verify the Laurent property there is an easier method, invented by Hickerson and described in [48]. By iterating the map five times we obtain $\{e_n = p_n/q_n\}_{n=5}^{10}$, for polynomials p_n and monomials q_n in the initial values $\{e_n\}_{n=1}^{5}$. As p_5 is prime to p_n for all $n \in \{6, 7, 8, 9, 10\}$ the recurrence (72) satisfies the Laurent property.

We remark that a reduction of order, by introducing the variable e_{n+1}/e_{n-1}, is apparent, however, this does not preserve Laurentness. Furthermore we should mention that the fact that the rational recurrence (72) with initial values (73) produces a polynomial sequence does not follow from the Laurent property of (72). One needs a strong Laurent property such as given in [32] for Somos sequences.

5.2. From the third order DTKQ equation to a sixth order Laurent recurrence with five terms, with coefficients that are periodic with period 8

Taking $N = 3$ in Equation (67), this gives the third order DTKQ equation,

$$u_{n+3} = \frac{\phi}{u_{n+1} u_{n+2}} - u_n - u_{n+1} - u_{n+2}.$$

This equation has two first integrals. Using one of them it should be possible to reduce it to a second order map of QRT-type. However, let us proceed with the third order map as given. Homogenising yields,

$$\begin{aligned} a_{n+3} &= \phi b_{n+1}^2 b_{n+2}^2 b_n - a_{n+1} a_{n+2} a_n b_{n+1} b_{n+2} - a_{n+1}^2 a_{n+2} b_{n+2} b_n - a_{n+1} a_{n+1}^2 b_{n+1} b_n, \\ b_{n+3} &= b_{n+1} b_{n+2} b_n a_{n+1} a_{n+2}. \end{aligned} \tag{75}$$

If we choose $\{a_n = u_n, b_n = 1\}_{n=1}^{3}$ then all a_n and b_n are polynomials in the initial variables u_1, u_2, u_3 and parameter ϕ. A sequence of polynomials $\{k_n\}_{n=1}^{\infty}$ is defined by:

$$a_n = \begin{cases} k_n & \text{if } n < 5, \\ k_3k_5 & \text{if } n = 5, \\ k_2^{m_n^a(k_2)} \left(\prod_{i=3}^{n-3} k_i^{\mathfrak{m}_{n-i-2}}\right) k_{l-3}k_{l-2}k_l & \text{if } n > 5, \end{cases}$$
$$b_n = \begin{cases} 1 & \text{if } n < 4, \\ k_2k_3 & \text{if } n = 4, \\ k_2^{m_l^b(k_2)} \left(\prod_{i=3}^{n-3} k_i^{\mathfrak{m}_{n-i-2}}\right) k_{n-2}^2 k_{n-1} & \text{if } n > 4, \end{cases} \tag{76}$$

where $\{\mathfrak{m}_n\}_{n=1}^{\infty}$ is the integer sequence defined by $\mathfrak{m}_1 = 4$, $\mathfrak{m}_2 = 13$, $\mathfrak{m}_3 = 37$ and $\mathfrak{m}_n = 2\mathfrak{m}_{n-1} + 2\mathfrak{m}_{n-2} + \mathfrak{m}_{n-3}$. In this case the ultra-discretization of the homogenized system does not give us a sharp bound on the multiplicities $m_n^a(k_2)$ and $m_n^b(k_2)$. By using prime numbers as initial values we were able to iterate the map (75) a little further than usual, which led us to formulate the following.

Conjecture 9: *The difference of the multiplicities of k_2 in a_n and b_n is periodic, we have $m_n^a(k_2) - m_n^b(k_2) = \zeta_n$, with $\zeta_{\,mod\,8} = [0, 1, 0, -1, -1, 2, -1, -1]$.*

Assuming the conjecture, from (76), it follows that

$$\frac{a_n}{c_{n-3}c_n g_n} = k_2^{\max(0,\zeta_n)} \quad \text{and} \quad \frac{b_n}{c_{n-2}c_{n-1}g_n} = k_2^{\max(0,-\zeta_n)}$$

are polynomial sequences in k_2. Using these functions we find the following theorem

Theorem 10: *The polynomials $k_{n>1}$, as defined by (76), satisfy*

$$\epsilon_n \frac{k_{n-3}k_{n+1}}{k_{n-1}^2} + \epsilon_{n+1}\frac{k_{n-2}^2k_{n+1}^2}{k_{n-1}^2k_n^2} + \epsilon_{n+2}\frac{k_{n-2}k_{n+2}}{k_n^2} + \epsilon_{n+3}\frac{k_{n-2}k_{n+3}}{k_{n-1}k_{n+2}} = \frac{\phi}{\epsilon_{n+1}\epsilon_{n+2}}\frac{k_nk_{n+1}}{k_{n-1}k_{n+2}}, \tag{77}$$

with $\epsilon_n = u_2^{\zeta_n}$, $\{k_n = 1\}_{n=-1}^{2}$, $k_3 = u_3$ and $k_4 = \phi - u_2u_3(u_1 + u_2 + u_3)$.

The Laurentness of (77) can be verified as before, this time the iteration has to be repeated for every congruence class n mod 8. We will discuss a generalization of the Hickerson method which includes periodic coefficients in [35], cf. Section 4.3.

6. From a generalized Lyness equation to two quite different Laurent recurrences

In this section, we compare two choices for the parameters in the generalized Lyness equation,

$$w_{n+3}w_n = \mu + \nu w_{n+1} + w_{n+2}. \tag{78}$$

For $\mu = \nu = 1$ the sequences generated by this recurrence are 8-periodic. Equation (78) with $\nu = 1$ has vanishing algebraic entropy and its dynamics is rather well understood. For example, for $\mu > 0$, there are continua of initial conditions giving rise to even $2q$-periodic sequences for all but finitely many $q \in \mathbb{N}$ [6]. On the other hand, taking $\nu \neq 1$ the algebraic

Table 1. The multiplicities of q_n in a_{n+k-1} and b_{n+k-1}, from iterating the map (79).

k	1	2	3	4	5	6	7	8	9	10	11
$\bar{\mathfrak{m}}_k^a$	1	0	0	0	0	1	3	6	10	18	34
$\bar{\mathfrak{m}}_k^b$	0	0	0	1	1	2	3	5	10	18	34

entropy of the map does not vanish, in fact, the map is not confining. We are interested in the different recurrences satisfied by the divisors occuring in these distinct families of equations.

6.1. *Integrable case $\nu = 1$*

The homogenized map (78) reads

$$a_n = b_{n-3}q_n, \tag{79}$$
$$b_n = a_{n-3}b_{n-2}b_{n-1}, \tag{80}$$

where $q_n = a_{n-2}b_{n-1} + a_{n-1}b_{n-2} + \mu b_{n-2}b_{n-1}$. We take $\{a_i = w_i\}_{i=1}^3$ and $\{b_i = 1\}_{i=1}^3$. From (79), the ultra-discrete system for a lower bound on the multiplicities is:

$$\begin{aligned} \mathfrak{m}_n^a &= \mathfrak{m}_{n-3}^b + \min\left(\mathfrak{m}_{n-2}^b + \mathfrak{m}_{n-1}^b, \mathfrak{m}_{n-2}^a + \mathfrak{m}_{n-1}^b, \mathfrak{m}_{n-2}^b + \mathfrak{m}_{n-1}^a\right), \\ \mathfrak{m}_n^b &= \mathfrak{m}_{n-3}^a + \mathfrak{m}_{n-2}^b + \mathfrak{m}_{n-1}^b. \end{aligned} \tag{81}$$

Solving this system with initial values

$$\mathfrak{m}_{-1}^a = \mathfrak{m}_0^a = 0, \quad \mathfrak{m}_1^a = 1, \text{ and } \mathfrak{m}_{-1}^b = \mathfrak{m}_0^b = \mathfrak{m}_1^b = 0, \tag{82}$$

one finds the periodic difference

$$\mathfrak{m}_n^b - \mathfrak{m}_n^a = s_n, \qquad s_{\bmod 8} = [-1, 0, 0, 1, 1, 1, 0, 0]. \tag{83}$$

Combining (81)–(83) we obtain

$$\mathfrak{m}_{n+3}^a = \mathfrak{m}_n^a + \mathfrak{m}_{n+1}^a + \mathfrak{m}_{n+2}^a + r_n \quad \text{and} \quad \mathfrak{m}_{n+3}^b = \mathfrak{m}_n^b + \mathfrak{m}_{n+1}^b + \mathfrak{m}_{n+2}^b - s_n, \tag{84}$$

where $r_n = s_{n+2} + s_{n-1} - s_{n-3}$. These sequences describe the multiplicity of the initial values exactly. However, the multiplicity of the divisors q_n grow a bit faster. The sequences of multiplicities of q_n in a_{n+k-1} and b_{n+k-1} are denoted $\bar{\mathfrak{m}}_k^a$, $\bar{\mathfrak{m}}_k^b$. Their first 10 values are given in Table 1 and their tails coincide and can be expressed in terms of tribonacci numbers $\bar{\mathfrak{m}}_k = \bar{\mathfrak{m}}_{k-1} + \bar{\mathfrak{m}}_{k-2} + \bar{\mathfrak{m}}_{k-3}$ with $\bar{\mathfrak{m}}_0 = \bar{\mathfrak{m}}_1 = \bar{\mathfrak{m}}_2 = 1$ [50, A000213], i.e for all $k > 8$, $\bar{\mathfrak{m}}_k^a = \bar{\mathfrak{m}}_k^b = 2\bar{\mathfrak{m}}_{k-5}$. It is remarkable that q_n^{34} divides a_{n+10}, which can't be seen from the ultra-discrete system.

We express the polynomials a_n and b_n in terms of a sequence $\{z_k\}_{k=1}^{\infty}$, where z_n is defined as the quotient of a_n after division by powers of z_i with $i < n$, as follows,

$$a_n = \left(\prod_{i=1}^{3} z_i^{\mathfrak{m}^a_{n-i+1}}\right)\left(\prod_{i=4}^{n} z_i^{\bar{\mathfrak{m}}^a_{n-i+1}}\right), \quad b_n = \left(\prod_{i=1}^{3} z_i^{\mathfrak{m}^b_{n-i+1}}\right)\left(\prod_{i=4}^{n} z_i^{\bar{\mathfrak{m}}^b_{n-i+1}}\right). \tag{85}$$

Making explicit the common divisor, we write (for $n \geq 11$)

$$a_n = \left(\prod_{i=1}^{3} z_i^{\varepsilon_{n-i+1}}\right) z_{n-7} z_n g_n, \quad b_n = \left(\prod_{i=1}^{3} z_i^{\varsigma_{n-i+1}}\right) z_{n-4} z_{n-3} g_n, \tag{86}$$

with $\varepsilon_n = \max(0, -s_n)$, $\varsigma_n = \max(0, s_n)$ and,

$$g_n = \left(\prod_{i=1}^{3} z_i^{\mathfrak{m}^g_{n-i+1}}\right)\left(\prod_{i=4}^{n-5} z_i^{2\bar{\mathfrak{m}}_{n-i-4}}\right) \frac{z_{n-6}}{z_{n-7}}, \quad \mathfrak{m}^g_k = \mathfrak{m}^a_k - \varepsilon_k. \tag{87}$$

Substituting (86) in (80), we find

$$\frac{g_n}{g_{n-3} g_{n-2} g_{n-1}} = \prod_{i=1}^{3} z_i^{\varepsilon_{n-i-2}+\varsigma_{n-i-1}+\varsigma_{n-i}-\varsigma_{n-i+1}} z_{n-5}^2 z_{n-6} z_{n-10}, \tag{88}$$

which is satisfied by (87). By substituting (86) in (79) and using (88),we find:

$$\begin{aligned} z_{n+3} z_{n-2} z_{n-7} = \mu & \prod_{i=1}^{3} z_i^{\varsigma_{n-i-2}-\varepsilon_{n-i+1}+\varsigma_{n-i+1}-\varepsilon_{n-i-2}} z_{n-1} z_{n-2} z_{n-3} \\ & + \prod_{i=1}^{3} z_i^{\varepsilon_{n-i-1}+\varsigma_{n-i-2}-\varepsilon_{n-i+1}+\varsigma_{n-i+1}-\varsigma_{n-i-1}-\varepsilon_{n-i-2}} z_{n-1} z_{n+1} z_{n-6} \\ & + \prod_{i=1}^{3} z_i^{\varepsilon_{n-i}+\varsigma_{n-i-2}-\varepsilon_{n-i+1}+\varsigma_{n-i+1}-\varsigma_{n-i}-\varepsilon_{n-i-2}} z_{n+2} z_{n-3} z_{n-5}. \end{aligned} \tag{89}$$

Express the coefficients in terms of

$$\delta_{\bmod 8} = [0, 1, 0, 1, 0, 0, 0, 0], \tag{90}$$

we arrive at the following theorem.

Theorem 11: *The polynomials $z_{n\geqslant 4}$, as defined by (85) are generated by*

$$z_{n+3} z_{n-2} z_{n-7} = \kappa_n z_{n-1} z_{n-2} z_{n-3} + \tau_n z_{n-1} z_{n+1} z_{n-6} + \sigma_n z_{n+2} z_{n-3} z_{n-5}, \tag{91}$$

where

$$\kappa_n = \mu \prod_{i=1}^{3} w_i^{\delta_{n-i}+\delta_{n-i+1}}, \quad \tau_n = \prod_{i=1}^{3} w_i^{\delta_{n-i+3}} \quad \textit{and} \quad \sigma_n = \prod_{i=1}^{3} w_i^{\delta_{n-i+6}},$$

from initial values $\{z_i = 1\}_{i=-6}^{3}$.

Therefore, the fact that $\{z_n\}_{n=1}^{\infty}$ is a sequence of polynomials is explained by the Laurent property of (91).

6.2. Non-integrable case $\nu \neq 1$

In this case the multiplicities of all divisors are given by the ultra-discrete system (81) with initial values (82), which yields (84). No surprising factorization occurs. Hence we get

$$a_n = \prod_{i=1}^{n} z_i^{\mathfrak{m}^a_{n-i+1}} \quad \text{and} \quad b_n = \prod_{i=1}^{n} z_i^{\mathfrak{m}^b_{n-i+1}}, \tag{92}$$

for all $n > 0$. Thus we have $g_n = \prod_{i=1}^{n} z_i^{\mathfrak{m}^g_{n-i+1}}$, where $\mathfrak{m}^g_n$ is given by (87) and we may write

$$a_n = \prod_{i=1}^{n} z_i^{\varepsilon_{n-i+1}} g_n \quad \text{and} \quad b_n = \prod_{i=1}^{n} z_i^{\varsigma_{n-i+1}} g_n. \tag{93}$$

By substituting (93) in (80), we find:

$$\frac{g_n}{g_{n-3}g_{n-2}g_{n-1}} = \prod_{i=1}^{n-3} z_i^{\varepsilon_{n-i-2}+\varsigma_{n-i-1}+\varsigma_{n-i}-\varsigma_{n-i+1}}. \tag{94}$$

Moreover, by substituting (93) in (79) and using (94), we find:

$$\begin{aligned} z_n = \mu \prod_{i=1}^{n-1} z_i^{\varsigma_{n-i-2}-\varepsilon_{n-i+1}-\varepsilon_{n-i-2}+\varsigma_{n-i+1}} \\ + \nu \prod_{i=1}^{n-1} z_i^{\varepsilon_{n-i-1}+\varsigma_{n-i-2}-\varepsilon_{n-i+1}-\varepsilon_{n-i-2}-\varsigma_{n-i-1}+\varsigma_{n-i+1}} \\ + \prod_{i=1}^{n-1} z_i^{\varepsilon_{n-i}+\varsigma_{n-i-2}-\varepsilon_{n-i+1}-\varepsilon_{n-i-2}-\varsigma_{n-i}+\varsigma_{n-i+1}} \end{aligned} \tag{95}$$

Expressing the result in terms of (90), we obtain the following theorem.

Theorem 12: *The nth term in the sequence* $\{z_k\}_{k=1}^{\infty}$ *is given by the polynomial expression*

$$z_n = \mu \left(\prod_{i=1}^{n-1} z_i^{\delta_{n-i-2}+\delta_{n-i-3}} \right) + \nu \left(\prod_{i=1}^{n-1} z_i^{\delta_{n-i}} \right) + \left(\prod_{i=1}^{n-1} z_i^{\delta_{n-i+3}} \right). \tag{96}$$

Note

1. Notation: a periodic function $p_{n+m} = p_n$ is defined by m values: with $p_{\bmod m} = [v_1, \ldots, v_m]$ we mean $p_n = v_{n \bmod m}$.

Acknowledgements

Both authors acknowledge useful discussions with Reinout Quispel. We thank Ralph Willox for bringing to our attention reference [40], and thank the referees for some useful remarks and additional references.

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

This research was supported by the Australian Research Council and by the La Trobe University Disciplinary Research Program in Mathematical and Computer Sciences.

References

[1] J. Alman, C. Cuenca, and J. Huang, *Laurent phenomenon sequences*, J. Algebr. Comb. (2015), pp. 1–45, doi:10.1007/s10801-015-0647-5, first online.
[2] V.I. Arnold, *Dynamics of complexity of intersections*, Bol. Soc. Bras. Mat. 21 (1990), pp. 1–10.
[3] M.P. Bellon and C.-M. Viallet, *Algebraic entropy*, Comm. Math. Phys. 204 (1999), pp. 425–437.
[4] S. Boukraa, J.-M. Maillard, and G. Rollet, *Integrable mappings and polynomial growth*, Phys. A 209 (1994), pp. 162–222.
[5] A.B. Buan, R. Marsh, M. Reineke, I. Reiten, and G. Todorov, *Tilting theory and cluster combinatorics*, Adv. Math. 204 (2006), pp. 572–618.
[6] A. Cima, A. Gasull, and V. Mañosa, *Dynamics of the third order Lyness' difference equation*, J. Difference Equ. Appl. 13 (2007), pp. 855–884.
[7] D.K. Demskoi, D.T. Tran, P.H. van der Kamp, and G.R.W. Quispel, *A novel nth order difference equation that may be integrable*, J. Phys. A: Math. Theor. 45 (2012), 135202 (10pp).
[8] S.B. Ekhad and D. Zeilberger, *How to generate as many Somos-like miracles as you wish*, J. Difference Equ. Appl. 20 (2014), pp. 852–858.
[9] G. Everest, G. Mclaren, and T. Ward, *Primitive divisors of elliptic divisibility sequences*, J. Number Theory 118 (2006), pp. 71–89.
[10] G. Everest, V. Miller, and N. Stephens, *Primes generated by elliptic curves*, Proc. Amer. Math. Soc. 132 (2003), pp. 955–963.
[11] G. Falqui and C.-M. Viallet, *Singularity, complexity and quasi-integrability of rational mappings*, Comm. Math. Phys. 154 (1993), pp. 111–125.
[12] V.V. Fock and A.B. Goncharov, *Cluster ensembles, quantization and the dilogarithm*, Ann. Sci. Éc. Norm. Supér 42 (2009), pp. 865–930.
[13] S. Fomin, *Cluster algebras portal*. (2016). Available at http://www.math.lsa.umich.edu/~fomin/cluster.html.
[14] S. Fomin and A. Zelevinsky, *Cluster algebras. I. Foundations*, J. Amer. Math. Soc. 15 (2002), pp. 497–529.
[15] S. Fomin and A. Zelevinsky, *The Laurent phenomenon*, Adv. Appl. Math. 28 (2002), pp. 119–144.
[16] A.P. Fordy and A.N.W. Hone, *Symplectic maps from cluster algebras*, SIGMA Symmetry Integrability Geom. Methods Appl. 7 (2011), pp. 1–12.
[17] A.P. Fordy and A.N.W. Hone, *Discrete integrable systems and Poisson algebras from cluster maps*, Comm. Math. Phys. 325 (2014), pp. 527–584.
[18] A.P. Fordy and R. Marsh, *Cluster mutation-periodic quivers and associated Laurent sequences*, J. Algebr. Combin. 34 (2011), pp. 19–66.
[19] D. Gale, *The strange and surprising saga of the Somos sequences*, Math. Intelligencer 13 (1991), pp. 40–42.

[20] Ch. Geiss, B. Leclerc, and J. Schröer, *Cluster algebras in algebraic Lie theory*, Transform. Groups 18 (2013), pp. 149–178.
[21] M. Gekhtman, M. Shapiro, and A. Vainshtein, *Cluster Algebras and Poisson Geometry*, American Mathematical Society, Providence, RI, 2010.
[22] B. Grammaticos and A. Ramani, *Integrability in a discrete world*, Chaos Solitons Fractals 11 (2000), pp. 7–18.
[23] K. Hamad, *Proving polynomial growth of degrees*, Masters thesis, La Trobe University, 2013.
[24] K. Hamad and P.H. van der Kamp, *From integrable equations to Laurent recurrences*, (2014). Available at arXiv:1412.5712v1.
[25] B. Hasselblatt and J. Propp, *Degree-growth of monomial maps*, Ergodic Theory Dynam. Systems 27 (2007), pp. 1375–1397.
[26] J. Hietarinta and C.-M. Viallet, *Singularity confinement and chaos in discrete systems*, Phys. Rev. Lett. 81 (1998), pp. 326–328.
[27] A.N.W. Hone, *Elliptic curves and quadratic recurrence sequences*, Bull. Lond. Math. Soc. 37(2) (2005), pp. 161–171.
[28] A.N.W. Hone, *Laurent polynomials and superintegrable maps*, SIGMA Symmetry Integrability Geom. Methods Appl. 3 (2007), 022 (18pp).
[29] A.N.W. Hone, *Sigma function solution of the initial value problem for Somos-5 sequences*, Trans. Amer. Math. Soc. 359 (2007), pp. 5019–5034.
[30] A.N.W. Hone, *Singularity confinement for maps with the Laurent property*, Phys. Lett. A 361 (2007), pp. 341–345.
[31] A.N.W. Hone and R. Inoue, *Discrete Painlevé equations from Y-systems*, J. Phys. A: Math. Theor. 47 (2014), 474007 (26pp).
[32] A.N.W. Hone and C. Swart, *Integrality and the Laurent phenomenon for Somos-4 and Somos-5 sequences*, Math. Proc. Cambridge Philos. Soc. 145 (2008), pp. 65–85.
[33] P.H. van der Kamp, *Somos-4 and Somos-5 are arithmetic divisibility sequences*, J. Difference Equ. Appl. (2015). Available at http://dx.doi.org/10.1080/10236198.2015.1113272.
[34] P.H. van der Kamp, *Growth of degrees of integrable mappings*, J. Difference Equ. Appl. 18 (2012), pp. 447–460.
[35] P.H. van der Kamp, K. Hamad, and G.R.W. Quispel, *Recursive factorisation of the symmetric QRT map*, in preparation.
[36] M. Kanki, J. Mada, T. Mase, and T. Tokihiro, *Irreducibility and co-primeness as an integrability criterion for discrete equations*, J. Phys. A: Math. Theor. 47 (2014), 465204 (15pp).
[37] T. Lam and P. Pylyavskyy, *Laurent phenomenon algebras*, (2012), Available at arXiv:1206.2611v2.
[38] X. Ma, *Magic determinants of Somos sequences and theta functions*, Discrete Math. 310 (2010), pp. 1–5.
[39] J.L. Malouf, *An integer sequence from a rational recursion*, Discrete Math. 110 (1992), pp. 257–261.
[40] T. Mase, *The Laurent phenomenon and discrete integrable systems*, RIMS Kôkyûroku Bessatsu B41 (2013), pp. 43–64.
[41] A. Nobe, *Ultradiscrete QRT maps and tropical elliptic curves*, J. Phys. A: Math. Theor. 41 (2008), 125205 (12pp).
[42] Y. Ohta, K.M. Tamizhmani, B. Grammaticos, and A. Ramani, *Singularity confinement and algebraic entropy: The case of the discrete Painlevé equations*, Phys. Lett. A 262 (1999), pp. 152–157.
[43] N. Okubo, *Discrete integrable systems and cluster algebras*, RIMS Kôkyûroku Bessatsu B41 (2013), pp. 25–41.
[44] A.J. van der Poorten and C.S. Swart, *Recurrence relations for elliptic sequences: Every Somos-4 is a Somos k*, Bull. Lond. Math. Soc. 38 (2006), pp. 546–554.
[45] G.R.W. Quispel, J.A.G. Roberts, and C.J. Thompson, *Integrable mappings and soliton equations II*, Phys. D 34 (1989), pp. 183–192.
[46] A. Ramani, B. Grammaticos, and J. Satsuma, *Bilinear discrete Painleve equations*, J. Phys. A: Math. Gen. 28 (1995), pp. 4655–4665.

[47] J. Roberts and D. Jogia, *Birational maps that send biquadratic curves to biquadratic curves*, J. Phys. A: Math. Theor. 48 (2015), 08FT02 (13pp).
[48] R. Robinson, *Periodicity of Somos sequences*, Proc. Amer. Math. Soc. 116 (1992), pp. 613–619.
[49] R. Shipsey, *Elliptic divisibility sequences*, PhD thesis, University of London, 2000.
[50] N.J.A. Sloane, *The online encyclopedia of integer sequences* (1964). Available at https://oeis.org/.
[51] M. Somos, *Problem 1470*, Crux Mathematicorum 15 (1989), p. 208.
[52] C.S. Swart, *Elliptic curves and related sequences*, PhD thesis, University of London, 2003.
[53] T. Tokihiro, D. Takahashi, J. Matsukidaira, and J. Satsuma, *From soliton equations to integrable cellular automata through a limiting procedure*, Phys. Rev. Lett. 76 (1996), pp. 3247–3250.
[54] A.P. Veselov, *Growth and integrability in the dynamics of mappings*, Comm. Math. Phys. 145 (1992), pp. 181–193.
[55] C.-M. Viallet, *On the algebraic structure of rational discrete dynamical systems*, J. Phys. A: Math. Theor. 48 (2015), 16FT01 (21pp).
[56] M. Ward, *Memoir on elliptic divisibility sequences*, Amer. J. Math. 70 (1948), pp. 31–74.
[57] R. Willox, T. Tokihiro, and J. Satsuma, *Darboux and binary Darboux transformations for the non-autonomous discrete KP equation*, J. Math. Phys. 38 (1997), pp. 6455–6469.
[58] A.V. Zabrodin, *A survey of Hirota's difference equations*, Theoret. Math. Phys. 113 (1997), pp. 1347–1392.
[59] http://www.maths.ed.ac.uk/~mwemyss/Somos5proof.pdf.