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Evolution of curvature invariants and lifting integrability

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Abstract

Given a geometry defined by the action of a Lie-group on a flat manifold, the Fels–Olver moving frame method yields a complete set of invariants, invariant differential operators, and the differential relations, or syzygies, they satisfy. We give a method that determines, from minimal data, the differential equations the frame must satisfy, in terms of the curvature and evolution invariants that are associated to curves in the given geometry. The syzygy between the curvature and evolution invariants is obtained as a zero curvature relation in the relevant Lie-algebra. An invariant motion of the curve is uniquely associated with a constraint specifying the evolution invariants as a function of the curvature invariants. The zero curvature relation and this constraint together determine the evolution of curvature invariants.

Invariantizing the formal symmetry condition for curve evolutions yield a syzygy between different evolution invariants. We prove that the condition for two curvature evolutions to commute appears as a differential consequence of this syzygy. This implies that integrability of the curvature evolution lifts to integrability of the curve evolution, whenever the kernel of a particular differential operator is empty. We exhibit various examples to illustrate the theorem; the calculations involved in verifying the result are substantial.

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Keywords: Moving frame; Curve; Curvature invariant; Lie group; Integrable evolution equation

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1. Introduction

Much has been written about the connection between geometry and integrability. Indeed, many integrable equations have been shown to describe the evolution of curvature invariants associated to a certain movement of curves in a particular geometric setting [2,6,11,17,20,24]. Some of the literature might give the impression that integrability arises from intrinsic properties of the underlying geometry. As was pointed out clearly in [20] this is not the case. However, it is easier to detect the integrability of the curvature evolution than that of the curve evolution, cf. [3].

Therefore the question arises whether integrability of the curvature evolution can be lifted to imply the integrability of the motion of the curve [12]. Hasimoto [6] showed that the invariant function $\psi = \kappa \exp(i \int \tau)$, where κ and τ are the curvature and torsion of a curve γ in Euclidean three space, evolves according to the integrable nonlinear Schrödinger equation

$$\psi_t = i\left(\psi_{xx} + \frac{1}{2}\psi|\psi|^2\right), \tag{1}$$

provided that the curve γ evolves according to the vortex filament equation

$$\gamma_t = \kappa e_3, \tag{2}$$

which relates the velocity of the curve to the bi-normal vector of the Serret–Frenet frame. Subsequently, Langer and Perline [12] translated the hierarchy of generalized symmetries of (1) to a hierarchy of commuting geometric curves, thereby establishing the integrability of Eq. (2) itself. Thus it seems that assigning to a curve its curvature functions gives rise to pairs of equivalent integrable equations [13].

In recent papers the lifting of integrability has been assumed. For example, in [2] it is remarked, "in view of the equivalence between the integrable equations for the curvature and the invariant motion, the motion law should also be integrable". Again, in [18] we find, "geometric evolutions would also be integrable in the sense that their associated curvature evolutions are, given that these determine the curve up to the action of the group". Moreover, in [10] it is stated, "We'll say such a flow is integrable if it induces a completely integrable system of PDE for curvature and torsion". However, such statements need justification, and more precision as to what aspects of integrability are meant. We will take the existence of symmetries to be the signature of integrability and we will demonstrate the lifting of integrability in this precise way.

The method of moving frames provides a powerful tool to study geometric properties, i.e., properties invariant under the action of a transformation group. This technique was introduced by Darboux, who studied curves and surfaces in Euclidean geometry, and was greatly developed by Cartan who used it in the context of generalizing Klein's Erlangen program. The formulation of the method by Fels and Olver [4,5] placed Cartan's constructions on a firm algebraic foundation. Their approach lead to new applications that would not have been envisioned by Cartan, such as to computer vision and numerical schemes that maintain symmetry [22].

The Fels–Olver moving frame method provides a generating set of invariants together with a maximal set of invariant differential operators and the differential relations, or *syzy-*

gies they satisfy. These data are all obtained with respect to a specific frame, which depends on a choice of submanifold which is transverse to the group orbits. One advantage of the method is its accessibility. In Section 2, we describe the ideas in the simplest possible language, the main tool being the chain rule of multi-variable calculus. More importantly, the method describes algorithmically what to do in any particular application and the calculations we require can be performed in a rigorous and straightforward way using symbolic software packages [1,16].

The purpose of this paper is twofold. Firstly, in Section 3, we present a method that provides the evolution equation for the curvature invariants of a curve, moving in a geometry which is given by the local action of a Lie-group on a manifold. The equation for the curvature invariants of a curve derives from a syzygy between sets of invariants. This is a zero-curvature condition in the relevant Lie-algebra and can be written in the form

$$\kappa_t = HI_t, \tag{3}$$

where H is a (matrix) differential operator, κ are the *curvature invariants* and I_t are the generating *evolution invariants* (see Section 2.3). Our contribution is to provide, from minimal data, the differential equations the frame satisfies using methods suitable for symbolic computation. These are obtained without solving for the moving frame, which, in general, is the central computational problem. The main result in Section 3, Theorem 8, is thus of independent interest.

The actual curvature evolution equation is obtained from relation (3) by specifying a constraint

$$I_t = F[\kappa],$$

where F is a (vector) function of the curvature invariant and their derivatives. This constraint is an invariant description of the evolution of the curve. Thus, one does not have to know the curvature invariants explicitly to obtain their evolution. However, there is a price to pay. From our point of view the filament equation is rather symbolic when one neither knows the curvature function κ, nor knows how to calculate the frame $\rho = (e_1, e_2, e_3)$. Within our approach there are two cases where an explicit form for the curve evolution may be obtained. Either one is able to solve the normalization equations (Section 2.2) for the frame, or one can use the Fels–Olver–Thomas replacement theorem (Section 2.3) to obtain the invariants in terms of classically known invariants of the group action. In either case the explicit expression for the curvature invariants provides the Miura transformation from the curve evolution to the curvature evolution.

The constraint might lead to an integrable equation for the curvatures. In Section 4, we answer the question whether integrability can be lifted: *Suppose that a curvature evolution is integrable, what can one say about the motion of the curve?* This is the second purpose of the paper. As we take the existence of infinitely many commuting symmetries to be the signature of integrability, our approach is to compare the symmetry conditions of both evolutions. The invariant form of the symmetry condition

$$D_{t_1}u_{t_2} - D_{t_2}u_{t_1} = 0$$

becomes a relation between the evolution invariants

$$C(I_{t_1}, I_{t_2}) = \mathcal{D}_{t_1} I_{t_2} + M_{t_1 t_2} - \mathcal{D}_{t_2} I_{t_1} - M_{t_2 t_1} = 0, \tag{4}$$

with *correction terms M* (see Section 2.3). For two invariant curve evolutions, specified by

$$I_{t_i} = F_i[\kappa], \ i = 1, 2, \tag{5}$$

the relation (4) gives a condition on the functions F_i, which is called the symmetry condition and is denoted as

$$C(I_{t_1}, I_{t_2})|_{I_{t_i}=F_i} = 0. \tag{6}$$

We show that the symmetry condition for curvature evolutions $\kappa_{t_i} = HF_i, \ i = 1, 2$ appears as a differential consequence of (6), that is

$$\mathcal{D}_{t_1}\kappa_{t_2} - \mathcal{D}_{t_2}\kappa_{t_1} - [\mathcal{D}_{t_1}, \mathcal{D}_{t_2}]\kappa = HC,$$

evaluated at the constraints (5). This implies that integrability does not necessarily lift from the curvature evolution to the curve evolution. However, most commonly studied integrable curvature equations are homogeneous polynomials or rational functions of the differential invariants. Since in these classes the kernel of the differential operator H is empty, pairs of integrable equations result, cf. [13]. In order to illustrate the scope of the theorem and the power of the method, we include, for several geometries, the explicit calculations that one would need to perform in the absence of the general result.

2. Moving frames à la Fels and Olver

In this section, we briefly describe the Fels and Olver moving frame formulation [4,5], in the language of undergraduate calculus. We give those details necessary to understand the proof of the main theorem of the next section, Theorem 8. We provide two expository examples which will be used in the sequel.

2.1. Group actions and prolongation

We are concerned with q functions u^α that depend on p variables x_i. New functions are obtained by differentiation and these will be denoted using a multi-index notation, e.g.

$$u^2_{112} = \frac{\partial^3}{\partial x_1^2 \partial x_2} u^2.$$

We consider all functions as independent and let them be the co-ordinates of a space M. Points in M will be denoted by $z = (x_1, \ldots, x_p, u^1, \ldots, u^q, u^1_1, \ldots)$. In other words, M is the jet bundle of the $(p + q)$-dimensional fibered manifold $X \times U$ where X is the space of independent variables and U is the space of dependent variables.

We will denote by $\mathcal{A}$ the ring of smooth functions on M, that depend on finitely many arguments. To indicate functional dependence of $f \in \mathcal{A}$ we simply write $f(z)$. The action of $\frac{\partial}{\partial x_i}$ extends to an action on $\mathcal{A}$ by the total differentiation operator

$$D_i = \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{q} \sum_K u^\alpha_{Ki} \frac{\partial}{\partial u^\alpha_K}. \tag{7}$$

We assume we are given a smooth left action of an r-dimensional Lie group G on the manifold $X \times U$. By prolongation we will get a left action on M, which is calculated using the chain rule of differentiation. The action $* : G \times M \to M$ satisfies $gh * z = g * h * z$. A right action on $\mathcal{A}$ is then given by $\bullet : G \times \mathcal{A} \to \mathcal{A}$; $g \bullet f(z) = f(g * z)$.

The image of a point under the action is denoted variously as

$$g * z = \tilde{z} = F(z, g)$$

or in terms of the co-ordinate functions as

$$g \bullet x_j = \tilde{x}_j = F_j(z, g), \quad g \bullet u^\alpha_K = \widetilde{u^\alpha_K} = F^\alpha_K(z, g)$$

The different notations are used to ease the exposition, depending on the context. The property of $*$ being a left action (or $\bullet$ being a right action acting component-wise on vector valued functions) is equivalent to $F(g * z, h) = F(z, hg)$.

The prolonged action is given explicitly by

$$g \bullet u^\alpha_{i\ldots j} = \widetilde{D}_i \ldots \widetilde{D}_j F^\alpha(z, g), \tag{8}$$

where

$$\widetilde{D}_i = \sum_{k=1}^{p} (\widetilde{D}x)_{ik} D_k \tag{9}$$

and the coefficients are obtained from the Jacobian matrix $(\tilde{D}x)_{ik} = (D_i \tilde{x}_k)^{-1}$.

The group elements $g \in G$ will be given in co-ordinates $g = (g_1, \ldots, g_r)$. With the group action comes an action of its Lie-algebra, obtained by formally expanding the group around its identity $\mathrm{e} \in G$. Let $g(\epsilon) \subset G$ be any one parameter subgroup of G such that $g(0) = \mathrm{e}$. Using the chain rule and Taylor's formula we obtain

$$\tilde{z} = z + \epsilon \sum_{i=1}^{r} \left. \frac{\partial \tilde{z}}{\partial g_i(\epsilon)} \right|_{g(\epsilon)=\mathrm{e}} \left. \frac{\mathrm{d} g_i(\epsilon)}{\mathrm{d}\epsilon} \right|_{\epsilon=0} + O(\epsilon^2).$$

where $g_i(\epsilon)$ are the co-ordinates of $g(\epsilon)$. Thus the infinitesimal generator of any one parameter subgroup is a linear combination of 'basic' infinitesimal generators. Their components are called the *infinitesimals* of the group action with respect to the i-th group parameter. A commonly used notation is

$$\xi_{j,i}(z) = \left.\frac{\partial \tilde{x}_j}{\partial g_i}\right|_{g=\mathrm{e}}, \quad \phi^\alpha_{,i}(z) = \left.\frac{\partial \tilde{u}^\alpha}{\partial g_i}\right|_{g=\mathrm{e}} \tag{10}$$

They classically depend only on (x, u). By iterative use of the chain rule, we may obtain recursion formula for the prolonged infinitesimals

$$\phi^\alpha_{K,i}(z) = \left.\frac{\partial \widetilde{z^\alpha_K}}{\partial g_i}\right|_{g=\mathrm{e}} \tag{11}$$

in terms of the $\xi_{j,i}$ and $\phi^\alpha_{,i}$. The recursion formula and its derivation can be found in textbooks on symmetries of differential equations cf. [[21], Theorem 2.36]. Further, they have been implemented in virtually every computer algebra system as part of Lie's algorithm to find symmetries of differential equations. A review of the software packages available has been given by W. Hereman in [9], (Vol. III, Chapter 13).

In the examples we will give names to every component of x and u. We also use the names of the components in the (multi)index instead of their numbers. For example in Example 1 we take $p = q = 2$. The components of x and u will be $x_1 = x$, $x_2 = t$, $u^1 = u$ and $u^2 = v$. And instead of u^2_{112} we write v_{xxt}.

Example 1. The Euclidean group $E(2) = SO(2) \ltimes \mathbb{R}^2$ acts on the variables (x, t, u, v) with $g = (\alpha, a, b)$ as

$$\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}.$$

leaving x and t invariant. Therefore we also have $\tilde{D}_x = D_x$, $\tilde{D}_t = D_t$ and hence the prolonged action is simply given by

$$g \bullet \begin{pmatrix} u_K \\ v_K \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} u_K \\ v_K \end{pmatrix}$$

The nonzero infinitesimals of the Euclidean group are

$$\phi^u_{K,\alpha}(z) = -v_K, \quad \phi^v_{K,\alpha}(z) = u_K, \quad \phi^u_{,a}(z) = 1, \quad \phi^v_{,b}(z) = 1.$$

Example 2. The second example is the group $SL(2)$ acting on the variables $(x, t, u(x, t))$ as $\tilde{t} = t$ and

$$\begin{pmatrix} \tilde{x} \\ \tilde{u} \end{pmatrix} = \begin{pmatrix} a & b \\ c & (1+bc)/a \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \tag{12}$$

where (a, b, c) are the co-ordinates of $g \in SL(2)$ near the identity $\mathrm{e} = (1, 0, 0)$. Formula (9) yields the following differential operators

$$\widetilde{D}_x = \frac{1}{a + bu_x} D_x, \quad \widetilde{D}_t = D_t - \frac{bu_t}{a + bu_x} D_x$$

From Eq. (8) it now follows that

$$\widetilde{u_x} = \frac{ac + u_x(1 + bc)}{a(a + bu_x)}, \quad \widetilde{u_{xx}} = \frac{u_{xx}}{(a + bu_x)^3}, \quad \widetilde{u_t} = \frac{u_t}{a + bu_x}$$

It can be checked that this is a right action. A table of infinitesimals is given below.

	x	u	u_x	u_{xx}	u_{xxx}
a	x	$-u$	$-2u_x$	$-3u_{xx}$	$-4u_{xxx}$
b	u	0	$-u_x^2$	$-3u_xu_{xx}$	$-4u_xu_{xxx} - 3u_{xx}^2$
c	0	x	1	0	0

The entry in the (i, y) place is the infinitesimal action corresponding to the i-th group parameter on the component y of z.

2.2. *Constructing a moving frame*

We use the Fels–Olver definition of a moving frame, and their approach to constructing them. This does not depend in any way on the presence of a frame bundle.

Definition 1. A *left moving frame* is a left G-equivariant map,

$$\rho : M \to G, \quad \rho(g * z) = g\rho(z),$$

and a *right moving frame* is a right G-equivariant map,

$$\rho : M \to G, \quad \rho(g * z) = \rho(z)g^{-1}.$$

A moving frame will exist if and only if the group action is free and regular. In our case the (sufficiently high) prolongation of the group action on M will be locally free provided the action on $X \times U$ is locally effective. We refer to [5] for the technical details.

The construction of a local moving frame in a neighborhood $\mathcal{U}$ proceeds as follows. Let $\mathcal{K} \subset \mathcal{U}$ be a sub-manifold which is transverse to the group orbits. We take $\mathcal{U}$ to be small enough so that each orbit intersects $\mathcal{K}$ at most once, cf. Fig. 1. Usually the cross-section $\mathcal{K}$ is the locus of a set of equations $\psi_k(z) = 0$, $k = 1, \ldots, r$, and then the so-called *normalization equations* for the frame are $\psi_k(\tilde{z}) = 0$, $k = 1, \ldots, r$. Solving these equations for the group parameters in terms of z yields a right frame.

Geometrically, the construction is as follows. For $z \in \mathcal{U}$, take $k \in \mathcal{K}$ and $h \in G$ such that $k = h * z$. The *right* moving frame $\rho : \mathcal{U} \to G$ is then defined by $\rho(z) = h$, and the *left* frame

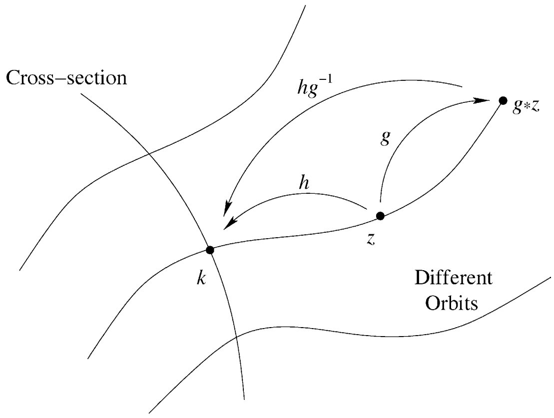

Fig. 1. Construction of a right moving frame using a cross-section.

by $\rho(z) = h^{-1}$. The right frame ρ is right equivariant since $\rho(g * z) = hg^{-1} = \rho(z)g^{-1}$ and a similar remark holds for the left frame, which is left equivariant.

One can think of the Fels–Olver moving frame as providing, locally, a trivialization of the manifold, i.e., when the frame is a right frame

$$\varphi : \mathcal{U} \to G \times \mathcal{K}, \quad z \to (\rho(z), \rho(z) * z)$$

is a trivialization of $\mathcal{U}$.

In the expository examples one can solve the normalization equations for the group parameters. In general, this will not be possible. However, to obtain the evolution of curvature invariants we do *not* need the frame to be known explicitly. This will be made clear in Section 3.

Example 1 (cont.). As normalization equations for the Euclidean group we choose

$$\widetilde{u} = 0, \quad \widetilde{v} = 0, \quad \widetilde{v_x} = 0. \tag{13}$$

These equations yield a right moving frame, mapping $z \in M$ to $\rho(z) \in G$ which has group parameters

$$\rho(z) = \left(-\arctan\left(\frac{v_x}{u_x}\right), -\frac{uu_x + vv_x}{\sqrt{u_x^2 + v_x^2}}, \frac{uv_x - vu_x}{\sqrt{u_x^2 + v_x^2}} \right).$$

A left moving frame is then given by the inverse $\rho^{-1}(z)$ which has parameters

$$\left(\arctan\left(\frac{v_x}{u_x}\right), u, v \right). \tag{14}$$

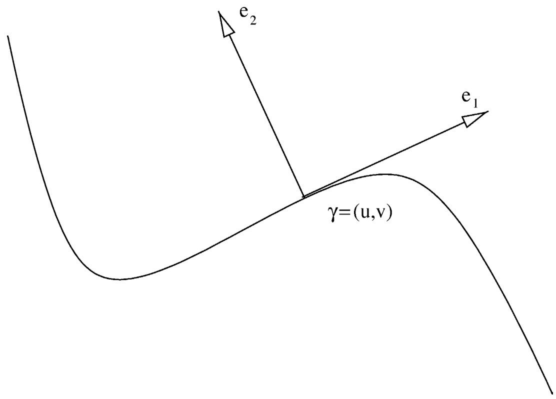

Fig. 2. An orthonormal frame attached to a curve in the plane.

Considering the rotational part of the group we note that at $g = \rho(z)$ the rows of

$$\begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} = \frac{1}{\sqrt{u_x^2 + v_x^2}} \begin{pmatrix} u_x & v_x \\ -v_x & u_x \end{pmatrix} \tag{15}$$

are the orthonormal vectors e_1 and e_2 attached to the curve $\gamma = (u, v)$ drawn in Fig. 2.

Note that we could also have started by defining the action of G on M to be the 'inverse' action

$$\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} u-a \\ v-b \end{pmatrix}.$$

The normalization equations (13) would then have yielded the moving frame (14) as a *right* moving frame.

Example 2 (cont.). For the matrix action of $SL(2)$ on (x, u), we can take the normalization equations

$$\widetilde{x} = 0, \quad \widetilde{u} = 1, \quad \widetilde{u_x} = 0. \tag{16}$$

The *right* frame is then given by

$$\rho(z) = \left(u, -x, \frac{u_x}{xu_x - u}\right). \tag{17}$$

2.3. Invariants and syzygies

In the sequel we assume that $\rho(z)$ is a right moving frame. The *normalized differential invariants* J_i, I^α, I^α_K are defined by evaluating the transformed dynamical variables on the frame. They are the components of

$$I(z) = \rho(z) * z.$$

Since

$$g \bullet (\rho(z) * z) = \rho(g * z) * (g * z) = \rho(z) g^{-1} * (g * z) = \rho(z) g^{-1} g * z = \rho(z) * z,$$

the function $I(z)$ is an invariant. The specific components of $I(z)$ are denoted

$$J_i = \tilde{x}_i|_{g=\rho(z)}, \quad I_K^\alpha = \widetilde{u_K^\alpha}\Big|_{g=\rho(z)}.$$

We can deal with these objects abstractly. However, explicit expressions for them in terms of the original variables can often be obtained. When the frame is known explicitly this is done by direct computation. The next theorem shows how the frame dependent invariants defined above may be related to known invariants, a procedure that will be illustrated in Example 4 of Section 4.

An important result is that any differential invariant is a function of the above invariants. This is a consequence of the *Fels–Olver–Thomas replacement theorem* ([5], Theorem 10.3), which states:

Theorem 2. *If $f(z) \in \mathcal{A}$ is an ordinary differential invariant then*

$$f(z) = f(I(z)).$$

This is true since in particular the action of $g = \rho(z) \in G$ leaves $f(z)$ invariant. As a corollary the set $\{J_i, I^\alpha, I_K^\alpha\}$ is a complete set of invariants.

The set of co-ordinates functions $\{u_K^\alpha\}$ can be obtained by acting with differentiation operators on the 'fundamental' set of dependent variables $\{u^\alpha\}$. Similarly the above complete set of invariants can be obtained by acting with invariant differential operators on a (finite) fundamental set of invariants.

Definition 3. A maximal set of *invariant operators* is defined by evaluating the transformed total differential operators on the frame. They are

$$\mathcal{D}_j = \widetilde{D}_j\Big|_{g=\rho(z)},$$

interpreted as derivations on $\mathcal{A}$.

One should be careful with the order of differentiation and substitution. In general we have that

$$\mathcal{D}_j I_K^\alpha = \widetilde{D}_j\Big|_{g=\rho(z)} \widetilde{u_K^\alpha}\Big|_{g=\rho(z)} \neq \tilde{D}_j \widetilde{u_K^\alpha}\Big|_{g=\rho(z)} = \widetilde{u_{Kj}^\alpha}\Big|_{g=\rho(z)} = I_{Kj}^\alpha.$$

This motivates the following definition.

Definition 4. The *correction terms* N_{ij} and M^{α}_{K} are defined by

$$\mathcal{D}_j J_i = \delta_{ij} + N_{ij}, \quad \mathcal{D}_j I^{\alpha}_{K} = I^{\alpha}_{Kj} + M^{\alpha}_{Kj}, \tag{18}$$

where δ_{ij} is the Kronecker delta.

It follows from their definition that the invariants are left unchanged by permutations within their index. The correction terms, however, are *not* invariant under permutations in their index.

Proposition 5. ([5], *Equation* (13.8)) *There exists an* $p \times r$ *correction matrix* $\mathbf{K}$ *such that*

$$N_{ki} = \sum_{j=1}^{r} \mathbf{K}_{ij} \xi_{k,j}(I(z)), \quad M^{\alpha}_{Ki} = \sum_{j=1}^{r} \mathbf{K}_{ij} \phi^{\alpha}_{K,j}(I(z)) \tag{19}$$

where j *is the index for the group parameters and* $r = \dim(G)$.

This result can be proved by application of the chain rule to $\tilde{D}_i I(z)$ evaluated at $g = \rho(z)$. It then follows that $\mathbf{K}$ is given by

$$\mathbf{K}_{ij} = \widetilde{D}_i \rho_j(\tilde{z})\Big|_{g=\rho(z)}$$

Its rows will take on additional significance in Section 3.

The matrix $\mathbf{K}$ can be calculated without explicit knowledge of the frame. All that is required are the normalization equations and the infinitesimal group action. Suppose the variables actually occurring in the $\psi_\lambda(z)$ are

$$\zeta_i = \begin{cases} x_{k_i}, & 1 \le i \le m, \\ u^{\alpha_i}_{K^i}, & m < i \le n. \end{cases}$$

Define $\mathbf{T}$ to be the invariant $p \times n$ total derivative matrix

$$\mathbf{T}_{ij} = \begin{cases} \delta_{k_j i}, & 1 \le j \le m, \\ I^{\alpha_j}_{K^j i}, & m < j \le n. \end{cases}$$

Also, let Φ denote the $r \times n$ matrix of invariant generators

$$\Phi_{ij} = \begin{cases} \xi_{k_j,i}(I), & 1 \le j \le m, \\ \phi^{\alpha_j}_{K^j,i}(I), & m < j \le n. \end{cases}$$

Furthermore, define $\mathbf{J}$ to be the invariant $n \times r$ transpose of the Jacobian matrix of ψ, that is

$$\mathbf{J}_{ij} = \begin{cases} \frac{\partial \psi_j(I)}{\partial J_{k_i}}, & 1 \le i \le m, \\ \frac{\partial \psi_j(I)}{\partial I^{\alpha_i}_{K^i}}, & m < i \le n. \end{cases}$$

Using the above defined matrices, which are easily calculated, the correction matrix can be obtained as follows.

Theorem 6. (*Olver*, [23]) *The correction matrix* $\mathbf{K}$, *which provides the error terms in the process of invariant differentiation in* Proposition 5, *is given by*

$$\mathbf{K} = -\mathbf{T}\mathbf{J}(\Phi\mathbf{J})^{-1}.$$

Proof. We compute the invariantization of the equations

$$D_i \psi_\lambda(\rho(z) * \zeta) = 0. \tag{20}$$

The invariantized normalization equations are functions of both the variables ζ_l and the co-ordinates of the frame $\rho_j(z)$. Since the latter depend on the first we have to be careful. We separate the different dependencies by writing $\psi_p(\rho(z) * \zeta) = \Psi_p(\zeta, \rho(z))$. Here the ψ's are functions of n variables, whereas the Ψ's depend on $n + r$ variables. Thus from Eq. (20) we obtain

$$\sum_{j=1}^{r} D_i \rho_j(z) \frac{\partial \Psi_\lambda(\zeta, \rho(z))}{\partial \rho_j(z)} + \sum_{l=1}^{n} D_i \zeta_l \frac{\partial \Psi_\lambda(\zeta, \rho(z))}{\partial \zeta_l} = 0.$$

We use the chain rule once more and write

$$\frac{\partial \Psi_\lambda(\zeta, \rho(z))}{\partial \rho_j(z)} = \sum_{l=1}^{n} \frac{\partial \rho(z) * \zeta_l}{\partial \rho_j(z)} \frac{\partial \psi_\lambda(\rho(z) * \zeta)}{\partial \rho(z) * \zeta_l}.$$

The theorem is proved by invariantization of the different terms, that is, replace z by $\tilde{z}$ (ζ by $\tilde{\zeta}$) and evaluate at $g = \rho(z)$. □

In a computer algebra environment, invariantization is achieved by substitution of the normalized invariants and simplification with respect to the normalization equations. For a discussion of the subtle issues that arise in this context we refer to [15]. In the meantime, we suppose that the simplification can be done by substitution of certain invariants that are highest with respect to a specified ordering. The set of such *highest normalized invariants* will be denoted by $\mathcal{H}$. Note that $\mathcal{H}$ is a subset of $\{\rho \bullet \zeta_i, i = 1, \dots, n\}$.

A classical theorem due to Tresse [25] states that all differential invariants can be obtained as functions of a finite number of invariants and their invariant derivatives. We have the following theorem.

Theorem 7. ([5], *Theorem* 13.4) *The set given by*

$$\{J^i, I^\alpha, I^\alpha_{Kj} | I^\alpha_K \in \mathcal{H}\} - \mathcal{H} \tag{21}$$

is a generating set of differential invariants.

This set is not necessary minimal, as will be shown in the examples. A major difference between the set $\{D_i, x_j, u^\alpha\}$ and the set of invariant differential operators with the generating invariants is the existence of nontrivial *syzygies*. Let I^α_J, I^α_L be two (generating) differential invariants, and indexes *K, M* are such that $I^\alpha_{JK} = I^\alpha_{LM}$. Then

$$\mathcal{D}_K I^\alpha_J - \mathcal{D}_M I^\alpha_L = M^\alpha_{JK} - M^\alpha_{LM} \tag{22}$$

is a *(fundamental) syzygy*, [5].

Example 1 (cont.). Since we have calculated a frame explicitly, the invariants can easily be expressed in terms of original variables. The components of $\rho(z) * z$ are the normalization equations

$$I^u = \rho(z) \bullet u = 0, \quad I^v = \rho(z) \bullet v = 0, \quad I^v_x = \rho(z) \bullet v_x = 0,$$

and the invariant functions

$$I^u_x = \sqrt{u_x^2 + v_x^2}, \quad I^u_K = \frac{u_x u_K + v_x v_K}{\sqrt{u_x^2 + v_x^2}}, \quad I^v_K = \frac{u_x v_K - v_x u_K}{\sqrt{u_x^2 + v_x^2}}.$$

The invariant operators are simply $\mathcal{D}_x = D_x$ and $\mathcal{D}_t = D_t$. Let us calculate the **K** matrix. We have

$$T = \begin{pmatrix} I^u_x & 0 & I^u_{xx} & I^v_{xx} \\ I^u_t & I^v_t & I^u_{xt} & I^v_{xt} \end{pmatrix}, \quad \Phi = \begin{pmatrix} 0 & 0 & I^u_x \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and hence

$$\mathbf{K} = -\begin{pmatrix} I^v_{xx}/I^u_x & I^u_x & 0 \\ I^v_{xt}/I^u_x & I^u_t & I^v_t \end{pmatrix}. \tag{23}$$

The correction terms are, for $\alpha \in \{u, v\}$, $i \in \{x, t\}$,

$$M^\alpha_i = -I^\alpha_i, \quad M^u_{Ki} = \frac{I^v_{xi} I^v_K}{I^u_x}, \quad M^v_{Ki} = -\frac{I^v_{xi} I^u_K}{I^u_x}.$$

The generating set of differential invariants is $\{I_x, I_t, I^u_t, I^u_x, I^v_t, I^v_{xx}, I^v_{xt}\}$ and we have the following fundamental syzygies.

$$\begin{aligned}
&\mathcal{D}_t I^u_x - \mathcal{D}_x I^u_t = -\tfrac{I^v_{xx} I^v_t}{I^u_x},\\
&I^v_{xt} - \mathcal{D}_x I^v_t = \tfrac{I^v_{xx} I^u_t}{I^u_x},\\
&\mathcal{D}_t I^v_{xx} - \mathcal{D}_x I^v_{xt} = \tfrac{I^v_{xx} I^u_{xt} - I^v_{xt} I^u_{xx}}{I^u_x}.
\end{aligned} \tag{24}$$

It is seen that the generating set is not minimal since the invariant I^v_{xt} may be removed using the second syzygy. For later reference, we note that the system may be written in the form (3), with $\mu = I^u_{xx}$, $\nu = I^u_x$,

$$\begin{pmatrix} \mu \\ \nu \end{pmatrix}_t = \begin{pmatrix} \mathcal{D}_x \frac{\mu}{\nu} + \frac{\mu}{\nu}\mathcal{D}_x + \frac{\mu \nu_x}{\nu^2} & \mathcal{D}_x^2 - \frac{\nu_x}{\nu}\mathcal{D}_x - \frac{\mu^2}{\nu^2} \\ \mathcal{D}_x & -\frac{\mu}{\nu} \end{pmatrix} \begin{pmatrix} I^u_t \\ I^v_t \end{pmatrix} \tag{25}$$

On the left hand side we have used subscript t to denote invariant time-differentiation and similarly on the right hand side ν_x denotes $\mathcal{D}_x \nu$ (which in this example is equal to $D_x \nu$).

Example 2 (cont.). Since we have only one dependent variable we will omit the upper index. Using the constructed moving frame we obtain the invariants

$$I_{xx} = \frac{u_{xx}}{(u - x u_x)^3}, \quad I_{xxx} = \frac{u u_{xxx} - x u_x u_{xxx} + 3 x u_{xx}^2}{(u - x u_x)^5}, \quad I_t = \frac{u_t}{u - x u_x}.$$

The invariant operators are found by substituting the frame into the transformed differentials (9). They are

$$\mathcal{D}_x = \frac{1}{u - x u_x} D_x, \quad \mathcal{D}_t = D_t + \frac{x u_t}{u - x u_x} D_x. \tag{26}$$

Using the matrices

$$T = \begin{pmatrix} 1 & 0 & I_{xx} \\ 0 & I^u_t & I_{xt} \end{pmatrix}, \quad \Phi = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we calculate the $\mathbf{K}$ matrix

$$\mathbf{K} = \begin{pmatrix} 0 & -1 & -I_{xx} \\ I_t & 0 & -I_{xt} \end{pmatrix}.$$

The generating set of invariants is $\{J_t, I_t, I_{xt}, I_{xx}\}$. Using the correction terms

$$M_{tx} = 0, \quad M_{xxt} = -3 I_t I_{xx}, \quad M_{xtx} = I_t I_{xx},$$

we find the syzygies

$$\begin{aligned} I_{xt} - \mathcal{D}_x I_t &= 0, \\ \mathcal{D}_t I_{xx} - \mathcal{D}_x I_{xt} &= -4 I_t I_{xx}. \end{aligned} \tag{27}$$

By eliminating I_{xt} we get

$$\mathcal{D}_2 I_{xx} = (\mathcal{D}_x^2 - 4I_{xx}) I_t,$$

which is an equation in the form (3).

3. Evolutions in the Lie-algebra

In Section 2 we have shown, using the Fels–Olver moving frame method, that the syzygies between the invariants can be obtained without solving for the frame. With regard to curves there are certain invariant functions that play a special role, the curvature invariants. In this section, we show how these can be obtained from the infinitesimals and the **K** matrix only. Subsequently the evolution of the curvature invariants is easily understood in terms of an evolution in the Lie-algebra of G.

Any sufficiently smooth curve on $X \times U$ will prolong to a curve in M. Suppose the curve is $s \mapsto z(s)$, and this lies in $\mathcal{U}$ where a moving frame is defined. Then the frame provides a curve in G, $s \mapsto G$, given by $s \mapsto \rho(z(s))$, see Fig. 3.

Consider the 1 + 1-dimensional case $(x, t) \mapsto z(x, t)$ where the two independent variables x and t are invariant and the operators D_x and D_t are thus invariant and commutative. When the group G is given as a matrix group, then the maps

$$x \mapsto \mathcal{Q}_x = (D_x \rho(z))\rho(z)^{-1}, \qquad t \mapsto \mathcal{Q}_t = (D_t \rho(z))\rho(z)^{-1}$$

are curves in the Lie-algebra $\mathfrak{g}$ of G, whose entries are invariants of the group action. The matrix $\mathcal{Q}_x$ is called the *curvature matrix* and its entries the *curvature invariants*. Viewing t to be 'time', the entries in $\mathcal{Q}_t$ will be called *evolution invariants*.

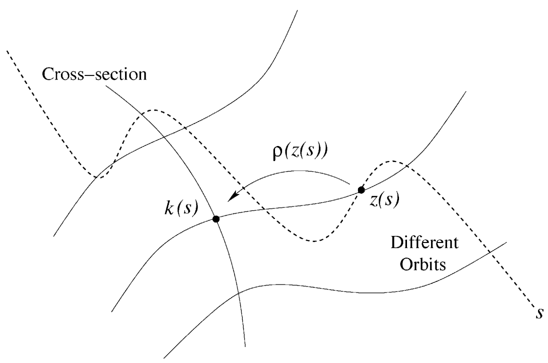

Fig. 3. A right moving frame for a curve parametrized by s.

We will show how to calculate these special invariants without knowing the moving frame $\rho(z)$ explicitly. The evolution of the curvature invariants is governed by a so called *zero-curvature* equation [7],

$$[D_t - \mathcal{Q}_t, D_x - \mathcal{Q}_x] = 0. \tag{28}$$

Here, the meaning of curvature in this phrase is not connected to that of the curve, but refers to the fact that the manifold is flat. In this section we treat a more general setting where the invariant operators do not necessarily commute. Thus the group may act non-trivially on the independent variables.

An n-dimensional matrix representation $\mathcal{R}$ of a group G is a map $G \to GL(\mathbb{R}^n)$ such that $\mathcal{R}(g)\mathcal{R}(h) = \mathcal{R}(gh)$. Note that this implies that $\mathcal{R}(\mathrm{e})$ is the identity matrix and $\mathcal{R}(g^{-1}) = \mathcal{R}(g)^{-1}$. By differentiating with respect to the group co-ordinates $g_1, \ldots g_r$ at the identity e, we obtain the infinitesimal generators

$$\mathfrak{a}_i = \left.\frac{\mathrm{d}\mathcal{R}(g)}{\mathrm{d}g_i}\right|_{g=\mathrm{e}}, \; i = 1, \ldots r$$

which span the Lie-algebra $\mathfrak{g}$ of G.

Let ϱ denote the matrix $\varrho = \mathcal{R}(\rho(z))$. We define the *curvature matrices*

$$\mathcal{Q}_i = (\mathcal{D}_i\varrho)\varrho^{-1}, \; i = 1, \ldots, p \tag{29}$$

The next theorem provides a new significance for the correction matrix $\mathbf{K}$; its rows are the co-ordinates of the curvature matrices, when expressed as a linear combination of the relevant basis of the Lie-algebra.

Theorem 8. *The curvature matrices $\mathcal{Q}_i$ can be constructed in the matrix representation of $\mathfrak{g}$ with basis $\{\mathfrak{a}_i\}$, using only the normalization equations and the infinitesimal action. Indeed*,

$$\mathcal{Q}_i = \sum_j \mathbf{K}_{ij}\mathfrak{a}_j$$

where $\mathbf{K}$ *is the correction matrix given in* Section 2.3.

Proof. Choose $g \in G$ arbitrary with $\tilde{z} = g * z$. On the one hand we have

$$\widetilde{D}_i\mathcal{R}(\rho(\tilde{z}))|_{g=\rho(z)} = \widetilde{D}_i(\mathcal{R}(\rho(z))\mathcal{R}(g^{-1}))|_{g=\rho(z)} = \widetilde{D}_i(\mathcal{R}(\rho(z)))\mathcal{R}(g)^{-1}|_{g=\rho(z)} = \mathcal{Q}_i$$

and on the other hand

$$\widetilde{D}_i\mathcal{R}(\rho(\tilde{z}))|_{g=\rho(z)} = \sum_{j=1}^{r} \widetilde{D}_i\rho_j(\tilde{z}) \left.\frac{\mathrm{d}\mathcal{R}(\rho(\tilde{z}))}{\mathrm{d}\rho_j(\tilde{z})}\right|_{g=\rho(z)} = \sum_{j=1}^{r} \mathbf{K}_{ij}\mathfrak{a}_j$$

since $\rho(\rho(z) * z) = \mathrm{e}$. □

The following proposition generalizes the zero-curvature Eq. (28) to include the case of non-commuting invariant differential operators and is essentially the structural formula for the Maurer-Cartan form.

Proposition 9. *The curvature matrices* (29) *satisfy the syzygy*

$$\mathcal{D}_j(\mathcal{Q}_i) - \mathcal{D}_i(\mathcal{Q}_j) = ([\mathcal{D}_j, \mathcal{D}_i]\varrho)\varrho^{-1} + [\mathcal{Q}_j, \mathcal{Q}_i]. \tag{30}$$

Proof.

$$\begin{aligned}\mathcal{D}_j(\mathcal{Q}_i) - \mathcal{D}_i(\mathcal{Q}_j) &= \mathcal{D}_j(\mathcal{D}_i(\varrho)\varrho^{-1}) - \mathcal{D}_i(\mathcal{D}_j(\varrho)\varrho^{-1}) \\ &= \mathcal{D}_j\mathcal{D}_i(\varrho)\varrho^{-1} - \mathcal{D}_i\mathcal{D}_j(\varrho)\varrho^{-1} + \mathcal{D}_i(\varrho)\mathcal{D}_j(\varrho^{-1}) - \mathcal{D}_j(\varrho)\mathcal{D}_i(\varrho^{-1}) \\ &= [\mathcal{D}_j, \mathcal{D}_i](\varrho)\varrho^{-1} + [\mathcal{Q}_j, \mathcal{Q}_i]\end{aligned}$$

as $\varrho\varrho^{-1} = 1$ implies $\mathcal{D}_k(\varrho^{-1}) = -\varrho^{-1}\mathcal{D}_k(\varrho)\varrho^{-1}$. □

The commutators of the invariant derivative operators can be calculated using only the **K** matrix and the infinitesimals of the group action. The following formula is taken from ([5], Equation 13.12). Denote the invariantized derivatives of the infinitesimals ξ by

$$\Xi^k_{li} = \widetilde{D}_i \xi_{k,l}(\tilde{z})|_{g=\rho(z)}.$$

Then we have

$$[\mathcal{D}_i, \mathcal{D}_j] = A^k_{ij}\mathcal{D}_k, \qquad A^k_{ij} = \sum_{l=1}^{r} \mathbf{K}_{jl}\Xi^k_{li} - \mathbf{K}_{il}\Xi^k_{lj}. \tag{31}$$

Remark 10. We will denote the curvature invariants that appear in the matrix $\mathcal{Q}_x$ by the vector κ. If the normalisation equations do not involve time-derivatives then it is always possible to rewrite the syzygy (30) in the form

$$\kappa_t = HI_t, \tag{32}$$

where H is a invariant matrix differential operator involving curvature invariants only. This is done by replacing I_{tKj} by $\mathcal{D}_{x_j} I_{tK} - M_{tKj}$ repeatedly.

Example 1 (cont.). The matrix

$$\mathcal{R}(g) = \begin{pmatrix} \cos\alpha & -\sin\alpha & a \\ \sin\alpha & \cos\alpha & b \\ 0 & 0 & 1 \end{pmatrix} \tag{33}$$

provides a representation of $E(2)$. The infinitesimal generators of the Lie-algebra are

$$\mathfrak{a}_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathfrak{a}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathfrak{a}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The **K** matrix (23) is used to calculate the curvature matrices

$$\mathcal{Q}_x = \begin{pmatrix} 0 & I^v_{xx}/I^u_x & -I^u_x \\ -I^v_{xx}/I^u_x & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{Q}_t = \begin{pmatrix} 0 & I^v_{xt}/I^u_x & -I^u_t \\ -I^v_{xt}/I^u_x & 0 & -I^v_t \\ 0 & 0 & 0 \end{pmatrix} \tag{34}$$

and their commutator

$$\mathcal{Q}_t\mathcal{Q}_x - \mathcal{Q}_x\mathcal{Q}_t = \begin{pmatrix} 0 & 0 & I^v_{xx}I^v_t/I^u_x \\ 0 & 0 & I^v_{xt} - I^v_{xx}I^u_t/I^u_x \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore from the matrix Eq. (30) we get the three equations

$$\begin{aligned} &\mathcal{D}_t\left(\frac{I^v_{xx}}{I^u_x}\right) - \mathcal{D}_x\left(\frac{I^v_{xt}}{I^u_x}\right) = 0, \\ &\mathcal{D}_t(I^u_x) - \mathcal{D}_x(I^u_t) = -\frac{I^v_{xx}I^v_t}{I^u_x}, \\ &\mathcal{D}_x(I^v_t) = I^v_{xt} - \frac{I^v_{xx}I^u_t}{I^u_x}, \end{aligned}$$

which are equivalent to the syzygies (24) obtained previously. With $\kappa = I^v_{xx}/I^u_x$, $\nu = I^u_x$ these can be written as

$$\begin{pmatrix} \kappa \\ \nu \end{pmatrix}_t = \begin{pmatrix} \mathcal{D}_x\frac{\kappa}{\nu} & \mathcal{D}_x\frac{1}{\nu}\mathcal{D}_x \\ \mathcal{D}_x & -\kappa \end{pmatrix} \begin{pmatrix} I^u_t \\ I^v_t \end{pmatrix}, \tag{35}$$

which should be compared with the system (25).

Recall that the rows of the rotational part of ρ are the vectors e_1 and e_2 along the curve, see Eq. (15). Suppose now that $\nu = 1$; since $I^u_x = e_1 \cdot D_x\gamma$ this corresponds to parameterizing the curve by arc-length. Expressing the evolution of κ in terms of I^v_t yields

$$\mathcal{D}_t\kappa = (\mathcal{D}^2_x + \kappa_x\mathcal{D}^{-1}_x\kappa + \kappa^2)I^v_t \tag{36}$$

The same equation is obtained from Eq. (25) since $I^u_x = 1$ implies that $I^u_{xx} = I^u_{xt} = 0$. One may recognize the recursion operator for the modified Korteweg–De Vries Eq. (47). Thus an integrable evolution equation is obtained when one imposes the constraint $I^v_t = \kappa_x$.

Example 2 (cont.). The infinitesimal generators in the Lie-algebra $\mathfrak{g}$ are given by

$$\mathfrak{a}_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; \mathfrak{a}_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; \mathfrak{a}_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Using the **K** matrix we get

$$\mathcal{Q}_x = \begin{pmatrix} 0 & -1 \\ -I_{xx} & 0 \end{pmatrix}, \; \mathcal{Q}_t = \begin{pmatrix} I_t & 0 \\ -I_{xt} & -I_t \end{pmatrix}.$$

By using Eq. (31), or the frame, we have $[\mathcal{D}_x, \mathcal{D}_t] = 2I_t\mathcal{D}_x$. Eq. (30) becomes

$$\mathcal{D}_t\mathcal{Q}_x - \mathcal{D}_x\mathcal{Q}_t = -2I_t\mathcal{Q}_x + [\mathcal{Q}_t, \mathcal{Q}_x]$$

and provides the syzygies (27).

4. Lifting integrability

In this section we answer the question whether integrability of a curvature evolution does lift to the motion of its curve. We take the existence of infinitely many generalized symmetries to be the signature of integrability. Therefore we aim to show, in general, that a symmetry of the curvature evolution gives rise to a symmetry of the curve evolution.

Suppose we have given two evolutions of curves,

$$u_{t_j} = P_j[u], \quad j = 1, 2.$$

Here $[u]$ denotes dependence on u as well as on x_i-derivatives of u. We have the following identity

$$(D_{t_1}D_{t_2} - D_{t_2}D_{t_1} - [D_{t_1}, D_{t_2}])u = 0. \tag{37}$$

We first look at (37) in the usual coordinates and then compare the calculations in the invariantised setting. The vanishing of the commutator $[D_{t_1}, D_{t_2}] = 0$ yields

$$D_{t_1}u_{t_2} - D_{t_2}u_{t_1} = 0, \tag{38}$$

which is the lowest order syzygy between time-derivatives of evolution variables. This identity in the differential algebra gives us a condition on the functions P_j. In practise one has to verify that

$$(D_{t_1}P_2)|_{u_{t_1}=P_1} - (D_{t_2}P_1)|_{u_{t_2}=P_2} = 0. \tag{39}$$

If it vanishes indeed, we say that the curve evolutions commute. This condition is called the symmetry condition. To evaluate the expressions one uses the trivial syzygies $D_K u = u_K$

(there are no correction terms) and the vanishing of the commutators

$$[D_{t_j}, D_{x_i}] = 0.$$

Next we will consider curve evolutions that are invariant under a given group action. The lowest order syzygy involving invariant time derivatives of the fundamental evolution invariants is

$$C(I_{t_1}, I_{t_2}) = \mathcal{D}_{t_1} I_{t_2} - \mathcal{D}_{t_2} I_{t_1} + M_{t_2 t_1} - M_{t_1 t_2} = 0. \tag{40}$$

Note that the correction terms may depend on the evolution invariants and space derivatives thereof, but not on their time-derivatives. Suppose that two invariant evolutions of a curve are given by

$$I_{t_i} = F_i[\kappa], \quad i = 1, 2, \tag{41}$$

where the F_i depend on the curvature invariants and their invariant derivatives. Let H be the matrix differential operator, see Remark 10, such that the time evolutions of κ, denoted $\kappa_{t_i} = \mathcal{D}_{t_i}\kappa$, are given by

$$\kappa_{t_i} = HF_i, \quad i = 1, 2. \tag{42}$$

The invariant symmetry condition is now given by

$$(\mathcal{D}_{t_1} F_2)|_{\kappa_{t_1}=HF_1} - (\mathcal{D}_{t_2} F_1)|_{\kappa_{t_2}=HF_2} + (M_{t_2 t_1} - M_{t_1 t_2})|_{I_{t_i}=F_i} = 0, \tag{43}$$

or, for short, by

$$C(I_{t_1}, I_{t_2})|_{I_{t_i}=F_i} = 0.$$

This is the same condition as condition (39), but now written in terms of invariants.

Theorem 11. *The symmetry condition for two curvature evolutions* (42) *is a differential consequence of the symmetry condition on the curve evolutions* (41). *We have that*

$$\mathcal{D}_{t_1} HI_{t_2} - \mathcal{D}_{t_2} HI_{t_1} - [\mathcal{D}_{t_1}, \mathcal{D}_{t_2}]\kappa = HC(I_{t_1}, I_{t_2}).$$

Proof. We look at both sides as differential expressions in the operators $\mathcal{D}_x$, $\mathcal{D}_{t_1}$ and $\mathcal{D}_{t_2}$ acting on function of κ, I_{t_1} and I_{t_2}. Note that for example $I_{t_1 t_2}$ does not appear in such an expression and every κ_{t_i} has been replaced by HI_{t_i}. We know that both sides vanish identically in the differential algebra of invariants. Since we can expand both identities into the form

$$H\mathcal{D}_{t_1} I_{t_2} + \ldots = 0.$$

where the dotted terms do not depend on $\mathcal{D}_{t_1} I_{t_2}$, both sides are equal as differential conditions on the invariant functions $I_{t_i} = F_i$. □

When the action of the Lie-group neither depends nor acts on the variables t_1 and t_2 and no evolution variables appear in the normalisation equations we can make the connection between the symmetry condition (38) and its invariantised form (40) more explicit. In this case the evolution invariants I_{t_i} will depend linearly on the original evolution variables u_{t_i}.

Let the Lie-group action be given by

$$\begin{aligned} \tilde{x}_i &= F_i(z, g), \quad i = 1, \ldots, p-2, \\ \tilde{t}_j &= t_j, \qquad\qquad j = 1, 2, \\ \tilde{u}^\alpha &= F^\alpha(z, g), \ \alpha = 1, \ldots, q. \end{aligned}$$

Suppose that the variables appearing in the normalisation equations are $\zeta_i = u^{\alpha_i}_{K^i}$ where t_1 and t_2 do not appear in K^i for any i. Then the $p \times p$ Jacobian matrix $D\tilde{x}$ is

$$D\tilde{x} = \begin{pmatrix} M & 0 & 0 \\ v_1 & 1 & 0 \\ v_2 & 0 & 1 \end{pmatrix}.$$

Here M is the $(p-2) \times (p-2)$ matrix $M = A + BC$ with $A_{ij} = \partial_{x_i} F_j$, $B_{i\alpha} = u^\alpha_i$ and $C_{\alpha j} = \partial_{u^\alpha} F_j$, and the v_k, $k = 1, 2$ are given by $(v_k)_j = \sum_\alpha u^\alpha_{t_k} C_{\alpha j}$. The inverse of this Jacobian matrix is given by

$$\widetilde{D}x = \begin{pmatrix} M^{-1} & 0 & 0 \\ -v_1 M^{-1} & 1 & 0 \\ -v_2 M^{-1} & 0 & 1 \end{pmatrix}.$$

Hence the transformed time-derivative operators are

$$\widetilde{D}_{t_k} = D_{t_k} - v_k M^{-1} D_x. \tag{44}$$

Applying such an operator to the transformed variables $\tilde{u}$ and then evaluating on the frame $g = \rho(z)$ gives us the matrix relating the evolution invariants and the evolution variables.

Proposition 12. *We have in this case*

$$I_{t_i} = R^{-1} u_{t_i}, \quad i = 1, 2,$$

where R is a matrix of functions of u^α_K with t_i not in the index K.

From Eq. (44) we also know that the invariant operators $\mathcal{D}_{t_j}$ equal D_{t_j} up to some correction term which is a linear operator in the $\mathcal{D}_x$ with coefficients that are linear in the I_{t_j}. Therefore the identities (38) and (40) are related by

$$R^{-1}(D_{t_1} u_{t_2} - D_{t_1} u_{t_2}) = \mathcal{D}_{t_1} I_{t_2} - \mathcal{D}_{t_2} I_{t_1} + M_{t_2 t_1} - M_{t_1 t_2}$$

by linearity of the derivations.

Theorem 11 implies that integrability does not necessarily lift from the curvature evolution to the curve evolution. However, most commonly studied integrable curvature equations are homogeneous polynomials or rational functions of the differential invariants. Since in these classes the kernel of the differential operator H is empty, pairs of integrable equations result, cf. [13]. In order to illustrate the scope of the theorem and the power of the method, we include, for several geometries, the explicit calculations that one would need to perform in the absence of the general result.

It can be seen in the examples that verifying the equation in Theorem 11 can involve substantial calculations. In particular, the fact that the operator H factors out of the left hand side is remarkable.

Another useful observation is that the explicit formulas for the curvature invariants provide a Miura-type transformation between the curve evolution and the curvature evolution. This will also be illustrated in the following examples.

Example 3. One example is provided by the group $SL(2)$ acting as $\tilde{x} = x, \tilde{t} = t$ and

$$\tilde{u} = \frac{au+b}{cu+d}, \quad ad - bc = 1.$$

The transformation relating the Schwarzian KDV Eq. (46) to the KDV Eq. (45) arises naturally in this context.

When we take $\tilde{u} = \widetilde{u_x} - 1 = \widetilde{u_{xx}} = 0$ as the normalization equations and take a, b, c as co-ordinates of the group, we have

$$\mathbf{K} = \begin{pmatrix} 0 & -1 & \frac{1}{2} I_{xxx} \\ -\frac{1}{2} I_{xt} & -I_t & \frac{1}{2} I_{xxt} \end{pmatrix}.$$

and, using the same basis for $\mathfrak{sl}(2)$ as in Example 2, we get

$$\mathcal{Q}_x = \begin{pmatrix} 0 & -1 \\ \frac{1}{2} I_{xxx} & 0 \end{pmatrix}, \quad \mathcal{Q}_t = \begin{pmatrix} -\frac{1}{2} I_{xt} & -I_t \\ \frac{1}{2} I_{xxt} & \frac{1}{2} I_{xt} \end{pmatrix},$$

and

$$[\mathcal{Q}_t, \mathcal{Q}_x] = \begin{pmatrix} \frac{1}{2}(I_{xxt} - I_t I_{xxx}) & I_{xt} \\ \frac{1}{2} I_{xt} I_{xxx} & \frac{1}{2}(I_t I_{xxx} - I_{xxt}) \end{pmatrix}.$$

Eq. (30) gives the following syzygies

$$\begin{aligned} &\mathcal{D}_x I_{xt} = I_{xxt} - I_t I_{xxx}, \\ &\mathcal{D}_x I_t = I_{xt}, \\ &\mathcal{D}_t I_{xxx} - \mathcal{D}_x I_{xxt} = I_{xxx} I_{xt}. \end{aligned}$$

We eliminate I_{xt} and I_{xtt} and denote $I_{xxx} = \kappa$ to get $\kappa_t = HI_t$, where

$$H = \mathcal{D}_x^3 + \kappa\mathcal{D}_x + \mathcal{D}_x\kappa$$

is one Hamiltonian operator of the KDV equation

$$\kappa_t = \kappa_{xxx} + 3\kappa\kappa_x, \tag{45}$$

which is famously integrable. When we impose the constraint $I_t = \kappa$ this implies that κ evolves according to KDV. Let us find out what the motion of the curve is. Using the moving frame,

$$\rho = \left(\frac{1}{\sqrt{u_x}}, \frac{-u}{\sqrt{u_x}}, \frac{u_{xx}}{2u_x\sqrt{u_x}}\right),$$

we obtain explicit expressions for the invariants,

$$I_t = \frac{u_t}{u_x}, \qquad \kappa = \frac{u_{xxx}}{u_x} - \frac{3}{2}\frac{u_{xx}^2}{u_x^2}.$$

Writing the constraint $I_t = \kappa$ in terms of the original co-ordinates we get the Schwarzian KDV equation,

$$u_t = u_{xxx} - \frac{3}{2}\frac{u_{xx}^2}{u_x}, \tag{46}$$

which is also well known to be integrable. Thus $\kappa = \{x; u\}$, the Schwarzian derivative, provides the Miura transformation between SKDV and KDV.

We compare the symmetry conditions on the different levels. Two different motions of the curve are given by different choices for the evolution invariant I_t as a function of the curvature invariant. The curve moves in different time-directions t_1, t_2 by $u_{t_i} = u_x F_i$, $i = 1, 2$. The condition on the functions F_1 and F_2 for these evolutions to commute is

$$\begin{aligned}
0 &= u_{t_1t_2} - u_{t_2t_1} \\
&= D_{t_2}(u_x F_1) - D_{t_1}(u_x F_2) \\
&= D_x(u_x F_2)F_1 + u_x D_{t_2} F_1 - D_x(u_x F_1)F_2 - u_x D_{t_1} F_2 \\
&= u_x(D_{t_2} F_1 - D_{t_1} F_2 - F_1 D_x F_2 + F_2 D_x F_1).
\end{aligned}$$

The symmetry condition for the curvature evolutions, $\kappa_{t_i} = HF_i$, to commute becomes

$$\begin{aligned}
0 &= \kappa_{t_1t_2} - \kappa_{t_2t_1} \\
&= D_{t_2} HF_1 - D_{t_1} HF_2 \\
&= HD_{t_2} F_1 + H(F_2)D_x F_1 + D_x H(F_2)F_1 - HD_{t_1} F_2 - H(F_1)D_x F_2 - D_x H(F_1)F_2 \\
&= H(D_{t_1} F_2 - D_{t_2} F_1 - F_1 D_x F_2 + F_2 D_x F_1),
\end{aligned}$$

where the last step can be verified by direct, albeit lengthy, computation, verifying Theorem 11.

The KDV equation has a recursion operator $\mathfrak{R} = HD_x^{-1}$. We can use this operator to write down the symmetries of SKDV. The constraint $I_t = D_x^{-1}\mathfrak{R}^{n-1}\kappa_x$ makes κ evolve according to a symmetry of KDV: $\kappa_{t'} = \mathfrak{R}^n\kappa_x$. Therefore we have

$$u_{t'} = u_x D_x^{-1}\mathfrak{R}^{n-1}\kappa_x,$$

where $\kappa = \{x; u\}$ is a symmetry of SKDV. Generating the symmetries this way is easier than by using the recursion operator for SKDV given in [26].

Example 2 (cont.). Although we do not know whether there is an integrable equation that arises as the curvature evolution of a curve moving in the geometric setting of the matrix action of $SL(2)$ on (x, u), still Theorem 11 implies that if it is in a class of equations where the kernel of $H = \mathcal{D}_x^2 - 4\kappa$ is empty, then the motion of its curve is integrable as well. The invariant evolution operators are, cf. Eq. (26),

$$\mathcal{D}_{t_i} = D_{t_i} + \frac{xu_{t_i}}{u - xu_x}D_x,$$

which commute with each other but not with $\mathcal{D}_x$. We impose constraints $I_{t_i} = F_i,\ i = 1, 2$ to describe the curve moving in different time t_i directions. The motions of the curves $u_{t_i} = (u - xu_x)F_i$ commute when

$$\begin{aligned}
0 &= u_{t_1t_2} - u_{t_2t_1}\\
&= D_{t_2}(u - xu_x)F_1 - D_{t_1}(u - xu_x)F_2\\
&= (u - xu_x)((D_{t_2} + xF_2D_x)F_1 - (D_{t_1} + xF_1D_x)F_2)\\
&= (u - xu_x)(\mathcal{D}_{t_2}F_1 - \mathcal{D}_{t_1}F_2).
\end{aligned}$$

Using the relation $[\mathcal{D}_x, \mathcal{D}_{t_i}] = 2F_i\mathcal{D}_x$ it can be verified that

$$\mathcal{D}_{t_2}(\mathcal{D}_x^2 - 4\kappa)I_{t_1} - \mathcal{D}_{t_1}(\mathcal{D}_x^2 - 4\kappa)I_{t_2} = (\mathcal{D}_x^2 - 4\kappa)(\mathcal{D}_{t_2}F_1 - \mathcal{D}_{t_1}F_2),$$

supporting Theorem 11.

Example 2 (cont.). It is also possible to have non commuting operators $\mathcal{D}_{t_i}$. Take *different* normalisation equations;

$$\tilde{x} = \tilde{u} = \widetilde{u_x} + 1 = 1.$$

Then

$$R = u - xu_x, \quad H = \mathcal{D}_x^2 - \mathcal{D}_x(\kappa) - 4\kappa,$$

with $\kappa = I_{xx}$. The commutators are

$$[\mathcal{D}_{t_1}, \mathcal{D}_{t_2}] = (I_{t_1}\mathcal{D}_x(I_{t_2}) - I_{t_2}\mathcal{D}_x(I_{t_1}))\mathcal{D}_x, \qquad [\mathcal{D}_x, \mathcal{D}_{t_i}] = (2I_{t_i} - \mathcal{D}_x(I_{t_i}))\mathcal{D}_x.$$

According to Theorem 11 we have

$$\begin{aligned}&\mathcal{D}_{t_1} HF_2 - \mathcal{D}_{t_2} HF_1 - (F_1\mathcal{D}_x(F_2) - F_2\mathcal{D}_x(F_1))\mathcal{D}_x(\kappa)\\ &\quad = H(\mathcal{D}_{t_1}F_2 - \mathcal{D}_{t_2}F_1 + F_1\mathcal{D}_x(F_2) - F_2\mathcal{D}_x(F_1))\end{aligned}$$

for arbitrary functions F_i. This can be verified by using the expressions for the operators in the original variables, or, by using the above commutation relations. There are algorithms available for processing differential systems given in terms of non-commutative derivations [8,14]. In general, when a frame cannot be constructed explicitly, this is the only option.

Example 1 (cont.). For the Euclidean action on the plane, after parameterizing by arc-length, we have obtained the syzygy $\kappa_t = \Re I_t^v$, with

$$\Re = D_x(D_x + \kappa D_x^{-1}\kappa),$$

cf. Eq. (36). The operator D_x is a Hamiltonian operator and $J = D_x + \kappa D_x^{-1}\kappa$ is a symplectic operator for the MKDV equation

$$\kappa_t = \kappa_{xxx} + \frac{3}{2}\kappa^2\kappa_x, \tag{47}$$

cf. [26].

However, to avoid the use of D_x^{-1}, we prefer to write the syzygy as $\kappa_t = HI_t^u$, where

$$H = D_x(\kappa + D_x\frac{1}{\kappa}D_x) \tag{48}$$

We consider two different curve evolutions given by $I_t^u = F_i,\ i = 1, 2$. The motion can be written in terms of frame vectors as

$$\gamma_{t_i} = F_i e_1 + \frac{1}{\kappa}D_x(F_i)e_2.$$

We know from $\mathcal{Q}_t$, see Eq. (34) that

$$D_{t_i}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = (\kappa + D_x\frac{1}{\kappa}D_x)(F_i)\begin{pmatrix} e_2 \\ -e_1 \end{pmatrix}.$$

Therefore the curves commute when

$$0 = D_{t_2}\gamma_{t_1} - D_{t_1}\gamma_{t_2} = Ce_1 + \frac{1}{\kappa}D_x(C)e_2,$$

where

$$C = D_{t_2}F_1 - D_{t_1}F_2 - F_2D_xF_1 + F_1D_xF_2 - \frac{D_x(F_1)D_x^2F_2 - D_x(F_2)D_x^2F_1}{\kappa^2}.$$

Using formula (48) for H and the evolutions of κ, it can be verified that indeed we have

$$D_{t_2}\kappa_{t_1} - D_{t_1}\kappa_{t_2} = H(C),$$

in agreement with Theorem 11.

Under the constraint $I_t^u = \frac{1}{2}\kappa^2$, or $I_t^v = D_x\kappa$, the curvature κ evolves according to the MKDV Eq. (47). If one imposes, in succession, the constraints

$$I_{t_m}^v = \mathfrak{R}^{m-1}\kappa_x, \; m = 1, 2, \ldots.$$

then their corresponding curvature evolutions are the symmetries of MKDV, namely $\kappa_{t_m} = \mathfrak{R}^m\kappa_x$. The curve evolutions defined by the constraints form a hierarchy as well. The lowest order ($m = 1$) constraint yields the following evolution for the curve

$$\gamma_t = \frac{1}{2}\kappa^2 e_1 + \kappa_x e_2, \tag{49}$$

which is called the planar filament equation. Using the recursion operator of MKDV to generate its higher symmetries is easier than the procedure given in [13].

Since the frame is known explicitly, it is easy to write equation (49) in terms of the original jet co-ordinates. By elimination of the second co-ordinate v using the constraint $u_1^2 + v_1^2 = 1$ we get for u the equation

$$u_t = u_{xxx} + \frac{3}{2}\frac{u_x u_{xx}^2}{1 - u_x^2} \tag{50}$$

These kind of scalar equations, i.e., third order equations linear in the highest derivative term, are classified with respect to integrability. Indeed, the above equation appears in the list ([19], equation 4.1.14). The explicit expression for the curvature, that is

$$\kappa = (u_x D_x - u_{xx})\sqrt{1 - u_x^2},$$

provides the Miura transformation that transforms Eq. (50) into MKDV. Yet other descriptions of the same geometric flow can be given, see Eqs. (53) and (54) in [2].

Example 4. We consider the motion of curves in 3 dimensional Euclidean space. The Cayley representation of $SO(3)$ is given by the matrix

$$\mathcal{R}(g) = \begin{pmatrix} g_0^2 + g_1^2 - g_2^2 - g_3^2 & 2(g_1g_2 - g_0g_3) & 2(g_1g_3 + g_0g_2) \\ 2(g_1g_2 + g_0g_3) & g_0^2 - g_1^2 + g_2^2 - g_3^2 & 2(g_2g_3 - g_0g_1) \\ 2(g_1g_3 - g_0g_2) & 2(g_2g_3 + g_0g_1) & g_0^2 - g_1^2 - g_2^2 + g_3^2 \end{pmatrix},$$

where $g_0^2 + g_1^2 + g_2^2 + g_3^2 = 1$. Let us write the vector of translation as $V(g) = (g_4, g_5, g_6)$ and define the action of the Euclidean group $E(3) = SO(3) \ltimes \mathbb{R}^3$ on $\gamma = (u, v, w)$ by

$$g * \gamma = \mathcal{R}(g)(\gamma - V(g)).$$

A representation of the group $E(3)$ is given by

$$\begin{pmatrix} \mathcal{R}(g) & V(g) \\ 0 & 1 \end{pmatrix}.$$

The normalization equations $I^u = I^v = I^w = I^v_x = I^w_x = I^w_{xx} = 0$ yield

$$\mathcal{Q}_x = \begin{pmatrix} 0 & \kappa & 0 & -I^u_x \\ -\kappa & 0 & \tau & 0 \\ 0 & -\tau & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{Q}_t = \begin{pmatrix} 0 & a & c & -I^u_t \\ -a & 0 & b & -I^v_t \\ -c & -b & 0 & -I^w_t \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where

$$\kappa = \frac{I^v_{xx}}{I^u_x}, \quad \tau = \frac{I^w_{xxx}}{I^v_{xx}}, \quad a = \frac{I^v_{xt}}{I^u_t}, \quad c = \frac{I^w_{xt}}{I^u_t}, \quad b = \frac{I^w_{xxt} I^u_x - I^w_{xt} I^u_{xx}}{I^u_x I^v_{xx}}. \tag{51}$$

Eq. (30) yields, after elimination of c,

$$\begin{pmatrix} \kappa \\ \tau \end{pmatrix}_t = \begin{pmatrix} D_x + \tau D_x^{-1} \tau & -\tau D_x^{-1} \kappa \\ -\kappa D_x^{-1} \tau & D_x + \kappa D_x^{-1} \kappa \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}.$$

Setting I^u_x to 1 and writing *a, b*, and *c* in terms of the generating evolution invariants yields

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \kappa + D_x \frac{1}{\kappa} D_x & -\tau \\ \tau + \frac{1}{\kappa}(\tau D_x + D_x \tau) \frac{1}{\kappa} D_x & \frac{1}{\kappa}(D_x^2 - \tau^2) \\ \frac{\tau}{\kappa} D_x & D_x \end{pmatrix} \begin{pmatrix} I^u_t \\ I^w_t \end{pmatrix}, \tag{52}$$

where we have eliminated $I^v_t = \frac{1}{\kappa} D_x I^u_t$.

The rotation part of our frame ρ, which has not been calculated explicitly, is related to the standard Serret–Frenet frame $(e_1, e_2, e_3)^{\mathrm{T}}$, that is, the $i -$ th row of $\mathcal{R}(\rho)$ equals e_i. Therefore we have

$$I^u_x = e_1 \cdot \gamma_x, \quad I^v_x = e_2 \cdot \gamma_x, \quad I^w_x = e_3 \cdot \gamma_x.$$

From the normalization equations we obtain $\gamma_x = I^u_x e_1$. Hence having $I^u_x = 1$ corresponds to x being arc-length. In terms of the Serret–Frenet frame the curve evolution is

$$\gamma_t = I^u_t e_1 + \frac{1}{\kappa} D_x(I^u_t) e_2 + I^w_t e_3.$$

In this equation I^u_t and I^w_t must be given in terms of κ and τ and their derivatives for the curve evolution to be invariant under the group action.

One integrable equation of this form is $\gamma_t = \kappa e_3$, called the filament equation, see [12]. This equation is equivalent to the constraints $I^u_t = 0$, $I^w_t = \kappa$. If γ evolves according to the filament equation, then κ and τ evolve according to

$$\begin{aligned} \kappa_t &= -\tfrac{1}{\kappa} D_x \tau \kappa^2 \\ \tau_t &= D_x\left(\tfrac{\kappa_x}{\kappa} + \tfrac{1}{2}\kappa^2 - \tau^2\right). \end{aligned} \tag{53}$$

An explicit form for the filament equation in terms of the original variables can be obtained from the explicit expressions for the frame dependent invariants κ, I^u_t and I^w_t. Without knowledge of the frame these can be obtained using the Fels–Olver–Thomas replacement rule from the known classical invariants. In this case the normalization equations yield

$$\begin{aligned} &|\gamma_x| = I^u_x, \\ &|\gamma_x \times \gamma_{xx}| = I^u_x I^v_{xx}, \\ &\gamma_x \cdot (\gamma_{xx} \times \gamma_{xxx}) = I^u_x I^v_{xx} I^w_{xxx}, \\ &|\gamma_t|^2 = I^{u\,2}_t + I^{v\,2}_t + I^{w\,2}_t, \\ &|\gamma_x \times \gamma_t|^2 = I^{u\,2}_x (I^{v\,2}_t + I^{w\,2}_t), \\ &\gamma_x \cdot (\gamma_{xx} \times \gamma_t) = I^u_x I^v_{xx} I^w_t. \end{aligned}$$

More directly one can use the well known explicit expression for the Serret–Frenet frame,

$$e_1 = \gamma_x, \quad e_2 = \frac{\gamma_{xx}}{|\gamma_{xx}|}, \quad e_3 = \frac{\gamma_x \times \gamma_{xx}}{\kappa}.$$

When the third co-ordinate w is eliminated using $|\gamma_x| = 1$ the filament equation $\gamma_t = \gamma_x \times \gamma_{xx}$, written in co-ordinates, is

$$\begin{aligned} u_t &= -\frac{v_{xx}(1-u_x^2) + u_{xx} u_x v_x}{\sqrt{1-u_x^2-v_x^2}}, \\ v_t &= \frac{u_{xx}(1-v_x^2) + v_{xx} v_x u_x}{\sqrt{1-u_x^2-v_x^2}}. \end{aligned} \tag{54}$$

The Miura transformation from Eq. (54) to Eq. (53) is provided by the explicit formulas for the curvature invariants

$$\kappa = \frac{\sqrt{u_{xx}^2 + v_{xx}^2 - (u_{xx}v_x - v_{xx}u_x)^2}}{\sqrt{1 - u_x^2 - v_x^2}},$$
$$\tau = \frac{u_{xx}v_x - v_{xx}u_x}{\sqrt{1 - u_x^2 - v_x^2}} - \frac{(u_{xxx}v_{xx} - v_{xxx}u_{xx})\sqrt{1 - u_x^2 - v_x^2}}{u_{xx}^2 + v_{xx}^2 - (u_{xx}v_x - v_{xx}u_x)^2}.$$

The operator H is now a 2×2 matrix. We have

$$\begin{pmatrix} \kappa_{t_i} \\ \tau_{t_i} \end{pmatrix} = H \begin{pmatrix} I^u_{t_i} \\ I^w_{t_i} \end{pmatrix},$$

with

$$H = \begin{pmatrix} D_x\kappa + (D_x^2 - \tau^2)\frac{1}{\kappa}D_x & -(D_x\tau + \tau D_x) \\ \tau D_x + D_x\tau + D_x\frac{1}{\kappa}(\tau D_x + D_x\tau)\frac{1}{\kappa}D_x & D_x\frac{1}{\kappa}(D_x^2 - \tau^2) \end{pmatrix}.$$

This operator is related to the Hamiltonian operator P given in ([18], Theorem 2), in the case of zero curvature, by interchanging the columns. We impose constraints

$$\begin{pmatrix} I^u_{t_i} \\ I^w_{t_i} \end{pmatrix} = \begin{pmatrix} F_i \\ G_i \end{pmatrix},\ i = 1, 2.$$

The corresponding curve t_i-evolutions commute when

$$0 = D_{t_2}\gamma_1 - D_{t_1}\gamma_2 = C_1 e_1 + \frac{1}{\kappa}D_x(C_1)e_2 + C_2 e_3,$$

where we have used $\mathcal{Q}_t$ and Eq. (52) to find

$$C_1 = F_2D_xF_1 - F_1D_xF_2 + G_1D_xG_2 - G_2D_xG_1 + 2\tfrac{\tau}{\kappa}(G_1D_xF_2 - G_2D_xF_1) + \tfrac{1}{\kappa^2}(D_x(F_1)D_x^2F_2 - D_x(F_2)D_x^2F_1),$$
$$C_2 = F_2D_xG_1 - F_1D_xG_2 + 2\tfrac{\tau}{\kappa^3}(D_x(F_2)D_x^2F_1 - D_x(F_1)D_x^2F_2) + \tfrac{\tau^2}{\kappa^2}(G_2D_xF_1 - G_1D_xF_2) + \tfrac{1}{\kappa^2}(D_x(F_2)D_x^2G_1 - D_x(F_1)D_x^2G_2).$$

Theorem 11 tells us that

$$D_{t_2}H\begin{pmatrix} F_1 \\ G_1 \end{pmatrix} - D_{t_1}H\begin{pmatrix} F_2 \\ G_2 \end{pmatrix} = H\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}.$$

To verify this requires quite a lengthy calculation. In particular, the integrability lifts from Eq. (53) to Eq. (54). A recursion operator for Eq. (53) was given in [18]. Similar to the planar case, this recursion operator can be used to write down the higher symmetries of the curve evolution (54) easily.

Example 5. Given an evolution equation, integrable or not, it is sensible to ask whether it arises as the curvature evolution for a curve moving in some geometry. However, this may happen in more than one way. We illustrate this by proving that any scalar evolution equation that allows a potential form arises both as the curvature evolution of a projected curve on the line and as the curvature evolution of a scaled curve on the line. By Theorem 11, these curves are integrable if the curvature evolution is integrable.

Suppose that a scalar equation for κ can be written as

$$\kappa_t = D_x F[\kappa], \tag{55}$$

for some F which is a function of κ and its x-derivatives. Then the equation has a *potential form*. The potential form of Eq. (55), obtained from the transformation $\kappa = \nu_x$, is given explicitly by

$$\nu_t = F[\nu_x].$$

For example, the potential form of Burgers' equation $\kappa_t = D_x(\kappa_x + \kappa^2)$ is

$$\nu_t = \nu_{xx} + \nu_x^2.$$

We first consider a curve $u(x)$ moving on the line where the geometry is given by $\tilde{x} = x$, $\tilde{u} = gu$. As a representation of the group $(\mathbb{R}^+, \cdot)$, we have $\mathcal{R}(g) = g$. We impose the normalization equation $\tilde{u} = 1$, which yields the right-moving frame $\rho = 1/u$. Among the invariants of the action we have $I_x = u_x/u$ and $I_t = u_t/u$. The curvature matrices are scalars, i.e., $\mathcal{Q}_x = -I_x$, $\mathcal{Q}_t = -I_t$. Since these commute, Eq. (30) yields $D_t I_x = D_x I_t$, which is of the form (55). We write $I_x = \kappa$ and $I_t = F[\kappa]$. Thus, Eq. (55) arises as the curvature evolution for the a scaled curve, whose evolution is given by

$$u_t = uF\left[\frac{u_x}{u}\right]. \tag{56}$$

Next we consider curves $u(x)$ moving on the line where the geometry is given by $\tilde{x} = x$, $\tilde{u} = u/(1 - gu)$. As a representation of the group $(\mathbb{R}, +)$ we have $\mathcal{R}(g) = e^g$. We impose the normalization equation $\tilde{u} = 1$. This yields the right-moving frame $\rho = (1 - u)/u$. Among the invariants of the action we have $I_x = u_x/u^2$ and $I_t = u_t/u^2$. The curvature matrices are $\mathcal{Q}_x = -I_x$ and $\mathcal{Q}_t = -I_t$, and we arrive to Eq. (55) again, however with different curvatures invariants $I_x = \kappa$ and $I_t = F[\kappa]$. Therefore, Eq. (55) also describes the curvature flow of a projective curve moving on the line where the evolution of the curve is given by

$$u_t = u^2 F\left[\frac{u_x}{u^2}\right]. \tag{57}$$

Note that Eq. (57) is equivalent to the potential form of the equation by the invertible transformation $v = -1/u$.

In the following table we present the curve evolutions whose curvatures evolve according to Burgers' equation (or the heat equation if $a = 0$), the Korteweg–De Vries equation and the nonlinear diffusion equation. Other equations which have a potential form include the modified KDV equation, the Sawada-Kotera equation and the Kaup-Kupershmidt equation. Their scaled and projective curve evolutions can be obtained directly from (56) and (57).

Curvature flow	scaled curve	projective curve
$\kappa_t = \kappa_{xx} + a\kappa_x\kappa$	$u_t = u_{xx} + (a-1)\frac{u_x^2}{u}$	$u_t = u_{xx} - 2\frac{u_x^2}{u} + \frac{a}{2}\frac{u_x^2}{u^2}$
$\kappa_t = \kappa_{xxx} + a\kappa_x\kappa$	$u_t = u_{xxx} - 3\frac{u_{xx}u_x}{u} + 2\frac{u_x^3}{u^2} + \frac{a}{2}\frac{u_x^2}{u}$	$u_t = u_{xxx} - 6\frac{u_{xx}u_x}{u} + 6\frac{u_x^3}{u^2} + \frac{a}{2}\frac{u_x^2}{u^2}$
$\kappa_t = D_x\frac{\kappa_x}{\kappa^2}$	$u_t = u_{xx}\frac{u^2}{u_x^2} - u$	$u_t = u_{xx}\frac{u^4}{u_x^2} - 2u^3$

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