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Duality for discrete integrable systems II

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Abstract

We generalise the concept of duality to lattice equations. We derive a novel 3-dimensional lattice equation, which is dual to the lattice AKP equation. Reductions of this equation include Rutishauser's quotient-difference (QD) algorithm, the higher analogue of the discrete time Toda (HADT) equation and its corresponding quotient–quotient-difference (QQD) system, the discrete hungry Lotka–Volterra system, discrete hungry QD, as well as the hungry forms of HADT and QQD. We provide three conservation laws, we conjecture the equation admits N-soliton solutions and that reductions have the Laurent property and vanishing algebraic entropy.

Keywords: integrable, 3D lattice equation, duality, conservation laws, solitons, Laurent property, algebraic entropy

(Some figures may appear in colour only in the online journal)

1. Introduction

Our aim in the present paper is threefold:

1. To generalise the concept of duality (introduced in [29] for ordinary difference equations) to lattice equations.
2. To use duality to derive the 3-dimensional (3D) lattice equation

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$$
\begin{aligned}
0 = {} & a_1 \left(\tau_{k-1,l+1,m+1}\tau_{k+1,l,m}\tau_{k+1,l,m+1}\tau_{k+1,l+1,m} - \tau_{k,l,m+1}\tau_{k,l+1,m}\tau_{k,l+1,m+1}\tau_{k+2,l,m}\right) \\
& + a_2 \left(\tau_{k,l+1,m}\tau_{k,l+1,m+1}\tau_{k+1,l-1,m+1}\tau_{k+1,l+1,m} - \tau_{k,l,m+1}\tau_{k,l+2,m}\tau_{k+1,l,m}\tau_{k+1,l,m+1}\right) \\
& + a_3 \left(\tau_{k,l,m+1}\tau_{k,l+1,m+1}\tau_{k+1,l,m+1}\tau_{k+1,l+1,m-1} - \tau_{k,l,m+2}\tau_{k,l+1,m}\tau_{k+1,l,m}\tau_{k+1,l+1,m}\right) \\
& + a_4 \left(\tau_{k,l,m}\tau_{k,l+1,m+1}\tau_{k+1,l,m+1}\tau_{k+1,l+1,m} - \tau_{k,l,m+1}\tau_{k,l+1,m}\tau_{k+1,l,m}\tau_{k+1,l+1,m+1}\right).
\end{aligned} \tag{1}
$$

3. To provide conservation laws for equation (1), to present reductions to two dimensional integrable systems, and to support our conjecture that equation (1) admits N-soliton solutions and its reductions have the Laurent property and vanishing algebraic entropy.

Most of currently known integrable 3D lattice equations are related to discretizations of the three continuous 3D Kadomtsev–Petviashvili equations called AKP, BKP and CKP. The lattice AKP equation,

$$A\tau_{k+1,l,m}\tau_{k,l+1,m+1} + B\tau_{k,l+1,m}\tau_{k+1,l,m+1} + C\tau_{k,l,m+1}\tau_{k+1,l+1,m} = 0, \tag{2}$$

was first derived by Hirota [18], and is also called the Hirota–Miwa equation [28]. The more general lattice BKP equation (also called the Miwa equation),

$$A\tau_{k+1,l,m}\tau_{k,l+1,m+1} + B\tau_{k,l+1,m}\tau_{k+1,l,m+1} + C\tau_{k,l,m+1}\tau_{k+1,l+1,m} + D\tau_{k,l,m}\tau_{k+1,l+1,m+1} = 0, \tag{3}$$

was first found by Miwa in [28]). The lattice CKP equation,

$$
\begin{aligned}
& \left(\tau_{k,l,m}\tau_{k+1,l+1,m+1} + \tau_{k+1,l,m}\tau_{k,l+1,m+1} - \tau_{k,l+1,m}\tau_{k+1,l,m+1} - \tau_{k,l,m+1}\tau_{k+1,l+1,m}\right)^2 \\
= {} & 4\left(\tau_{k,l,m}\tau_{k+1,l,m+1} - \tau_{k,l+1,m}\tau_{k,l,m+1}\right)\left(\tau_{k+1,l,m}\tau_{k+1,l+1,m+1} - \tau_{k+1,l+1,m}\tau_{k+1,l,m+1}\right)
\end{aligned} \tag{4}
$$

was first derived by Kashaev as a 3D lattice model associated with the local Yang–Baxter relation [22], and later was independently found by Schief [31] as a superposition principle for the continuous CKP equation. This equation is also formulated as a hyperdeterminant in [33].

The AKP equation is a bilinear equation on a six-point octahedral stencil (A_3 lattice). Equations of this type have been classified with respect to multi-dimensional consistency in [1]. The lattice BKP and CKP equations are both defined on an 8-point cubic stencil. However, whereas lattice BKP is bilinear, the lattice CKP is quartic and nonlinear. A nonlinear form of the AKP equation (quartic and defined on a 10-point stencil) was given in [13, equation (5.5)]. A quintic nonlinear non-potential lattice AKP equation was given in [11, equation (3.19)]. This equation is defined on a 10-point stencil [11, figure 3]. A quadrilinear 3D lattice equation related to the lattice BKP equation, defined on a 14-point stencil (D_3 lattice), is presented in [23, equation (24)]. Our equation (1), which we will obtain as a dual to the AKP equation (2), is a quadrilinear equation defined on the 14 point stencil depicted in figure 1.

To our knowledge equation (1) is new. Given that the number of known integrable 3D lattice equations is quite small, see [16, sections 3.9–3.10], any possible addition to this number would seem worthwhile pursuing.

The idea of duality for ordinary difference equations is as follows: given an ordinary difference equation (OΔE), $E = E(u_n, u_{n+1}, \ldots, u_{n+d}) = 0$, with an integral, $K[n] = K(u_n, u_{n+1}, \ldots, u_{n+d-1})$, the difference of the integral with its upshifted version factorises $K[n+1] - K[n] = E\Lambda$. The quantity Λ is called an integrating factor. The equation $\Lambda = 0$ is a dual equation to the equation $E = 0$, both equations share the same integral. If $E = 0$ has several integrals K_i, then a linear combination of them gives rise to a dual with parameters:

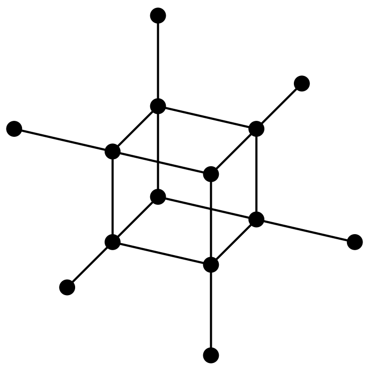

Figure 1. The 14-point stencil of equation (1).

$$\sum_i a_i K_i[n+1] - \sum_i a_i K_i[n] = E\left(\sum_i a_i \Lambda_i\right).$$

In [29] duals to $(d-1,-1)$-periodic reductions of the modified Korteweg–de Vries (mKdV) lattice equation are shown to be integrable maps, namely level-set-dependent mKdV maps. In [4] a novel hierarchy of maps is found by applying the concept of duality to the linear equation $u_n = u_{n+d}$, and $\lfloor \frac{d-1}{2} \rfloor$ integrals are provided explicitly. The integrability of these maps is established in [20]. We note that dual equations are not necessarily integrable, examples exist where the dual is not integrable [5]. In [21], the authors study several integrable 4th order maps and integrable maps that are dual to them.

Given a 2D lattice equation, $E = E(u_{k,l}, \dots, u_{k+d,l+e}) = 0$, instead of considering differences of integrals we now consider conservation laws:

$$P[k+1,l] - P[k,l] + Q[k,l+1] - Q[k,l] = E\Lambda.$$

Here the quantity Λ is called the characteristic of the conservation law. Again the equation $\Lambda = 0$ or a linear combination, $\sum_i a_i \Lambda_i = 0$, can be viewed as the dual equation to $E = 0$. The situation for 3D lattice equations is similar.

The structure of the paper is as follows. In section 2 we present a 3D lattice equation which is dual to the lattice AKP equation, and we provide a matrix formula which simultaneously captures four conservation laws for the AKP equation as well as three conservation laws for the dual AKP equation. In section 3 we show that these conservation laws give rise to quotients-difference formulations, in the same way that Rutishauser's quotient-difference (QD) algorithm [30] is a quotient-difference formulation of the discrete-time Toda equation [19]. In section 4 we provide 1-soliton and 2-solition solutions, and we provide evidence to support a conjectured form for the general N-soliton solution. In section 5 we provide evidence to support our conjecture that 2-periodic reductions to ordinary difference equations have the Laurent property. In section 6 we provide details of calculations which indicate that 2-periodic reductions of the dual AKP equation have quadratic growth. In section 7 we show that reductions of the dual AKP equation (1) to 2D lattice equations include the higher analogue of the discrete time Toda (HADT) equation and its corresponding quotient–quotient-difference (QQD) system [32], the discrete hungry Lotka–Volterra system, discrete hungry QD, as well as the hungry forms of HADT and QQD introduced in [3].

2. Derivation of a dual to the lattice AKP equation, and a matrix conservation law

Seven characteristics of conservation laws for the lattice AKP equation can be obtained from the results in [26]. We choose to only consider the parameter independent ones and we set all arbitrary functions equal to one. We will denote shifts in k using tildes, shifts in l by hats, and shifts in m by dots, e.g. $\tilde{\hat{\underaccent{\dot}{\tau}}} = \tau_{k+1,l+1,m-1}$. The four characteristics (denoted $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_7$ in [26]) are given by

$$W = \left(\frac{\underaccent{\tilde}{\hat{\dot{\tau}}}}{\hat{\dot{\tau}}\hat{\tau}\dot{\tau}} - \frac{\tilde{\tilde{\tau}}}{\tilde{\tau}\tilde{\hat{\tau}}\tilde{\dot{\tau}}},\ \frac{\underaccent{\hat}{\tilde{\dot{\tau}}}}{\tilde{\dot{\tau}}\tilde{\tau}\dot{\tau}} - \frac{\hat{\hat{\tau}}}{\hat{\tau}\hat{\dot{\tau}}\tilde{\hat{\tau}}},\ \frac{\underaccent{\dot}{\tilde{\hat{\tau}}}}{\tilde{\hat{\tau}}\tilde{\tau}\hat{\tau}} - \frac{\dot{\dot{\tau}}}{\dot{\tau}\tilde{\dot{\tau}}\hat{\dot{\tau}}},\ \frac{\tau}{\tilde{\tau}\hat{\tau}\dot{\tau}} - \frac{\tilde{\hat{\dot{\tau}}}}{\tilde{\hat{\tau}}\hat{\dot{\tau}}\tilde{\dot{\tau}}} \right).$$

One can now check the following matrix conservation law

$$\tilde{P} - P + \hat{Q} - Q + \dot{R} - R = V^T W, \tag{5}$$

where P, Q, R are the 3×4 matrices

$$P = \begin{pmatrix} -\frac{\tilde{\tau}\underaccent{\tilde}{\hat{\dot{\tau}}}}{\hat{\tau}\dot{\tau}} & 0 & 0 & -\frac{\tau\hat{\dot{\tau}}}{\hat{\tau}\dot{\tau}} \\ -\frac{\tilde{\tau}\underaccent{\tilde}{\hat{\tau}}}{\hat{\tau}\tau} & 0 & \frac{\dot{\tau}\underaccent{\dot}{\hat{\tau}}}{\hat{\tau}\tau} & 0 \\ -\frac{\tilde{\tau}\underaccent{\tilde}{\dot{\tau}}}{\dot{\tau}\tau} & \frac{\hat{\tau}\underaccent{\hat}{\dot{\tau}}}{\dot{\tau}\tau} & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & -\frac{\hat{\tau}\underaccent{\hat}{\tilde{\tau}}}{\tilde{\tau}\tau} & \frac{\dot{\tau}\underaccent{\dot}{\tilde{\tau}}}{\tilde{\tau}\tau} & 0 \\ 0 & -\frac{\hat{\tau}\underaccent{\hat}{\tilde{\dot{\tau}}}}{\tilde{\tau}\dot{\tau}} & 0 & -\frac{\tau\tilde{\dot{\tau}}}{\tilde{\tau}\dot{\tau}} \\ \frac{\tilde{\tau}\underaccent{\tilde}{\dot{\tau}}}{\dot{\tau}\tau} & -\frac{\hat{\tau}\underaccent{\hat}{\dot{\tau}}}{\dot{\tau}\tau} & 0 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & \frac{\hat{\tau}\underaccent{\hat}{\tilde{\tau}}}{\tilde{\tau}\tau} & -\frac{\dot{\tau}\underaccent{\dot}{\tilde{\tau}}}{\tilde{\tau}\tau} & 0 \\ \frac{\tilde{\tau}\underaccent{\tilde}{\hat{\tau}}}{\hat{\tau}\tau} & 0 & -\frac{\dot{\tau}\underaccent{\dot}{\hat{\tau}}}{\hat{\tau}\tau} & 0 \\ 0 & 0 & -\frac{\dot{\tau}\underaccent{\dot}{\tilde{\hat{\tau}}}}{\tilde{\tau}\hat{\tau}} & -\frac{\tilde{\hat{\tau}}\tau}{\tilde{\tau}\hat{\tau}} \end{pmatrix},$$

and V^T denotes the transpose of

$$V = \left(\hat{\dot{\tau}}\tilde{\tau},\ \tilde{\dot{\tau}}\hat{\tau},\ \tilde{\hat{\tau}}\dot{\tau} \right).$$

Denoting two vectors of coefficients by $X = (A, B, C)$ and $Y = (a_1, a_2, a_3, a_4)$, we have that $XV^T = 0$ represents the AKP equation (2) and the equation $WY^T = 0$ is equivalent to equation (1).

Hence, pre-multiplying (5) with X gives four conservation laws for the lattice AKP equation, and post-multiplying (5) with Y^T yields three conservation laws for equation (1). Thus the lattice AKP equation and equation (1) are dual to each other.

3. Corresponding quotients-difference systems

The lattice AKP equation and the dual AKP equation can each be written as a system of one difference equation combined with a number of quotient equations.

Let us introduce variables

$$p = \frac{\tau\hat{\dot{\tau}}}{\hat{\tau}\dot{\tau}}, \qquad q = \frac{\tau\tilde{\dot{\tau}}}{\tilde{\tau}\dot{\tau}}, \qquad v = \frac{\tilde{\hat{\tau}}\tau}{\tilde{\tau}\hat{\tau}}.$$

Note that $p = -P_{14}$, $q = -Q_{24}$ and $v = -R_{34}$. Hence, the fourth conservation law for the AKP equation can be written as the difference equation

$$A(\tilde{p} - p) + B(\hat{q} - q) + C(\dot{v} - v) = 0. \tag{6}$$

Taking logarithms, we find

$$\begin{aligned} \ln(p) &= c + \hat{\dot{c}} - \hat{c} - \dot{c} = (1 - \hat{S})(1 - \dot{S})c \\ \ln(q) &= c + \tilde{\dot{c}} - \tilde{c} - \dot{c} = (1 - \tilde{S})(1 - \dot{S})c \\ \ln(v) &= c + \tilde{\hat{c}} - \tilde{c} - \hat{c} = (1 - \tilde{S})(1 - \hat{S})c \end{aligned}$$

where $c = \ln(\tau)$, capital $\tilde{S}$ denotes the shift operator in k (and similarly $\hat{S}$ and $\dot{S}$ represent shifts in l resp. m), and 1 is the identity. This gives $(1 - \tilde{S})\ln(p) = (1 - \hat{S})\ln(q) = (1 - \dot{S})\ln(v)$, which can be written in quotient form,

$$\frac{\tilde{p}}{p} = \frac{\hat{q}}{q} = \frac{\dot{v}}{v}. \tag{7}$$

As (7) contains only two independent equations the system of equations for p, q, v defined by (6) and (7) can be referred to as a QQD-system.

Similarly, we can write the dual AKP equation in variables

$$u = \frac{\tilde{\tau}\dot{\tilde{\tau}}}{\tau\dot{\tau}}, \quad z = \frac{\hat{\tau}\dot{\hat{\tau}}}{\tau\dot{\tau}}, \quad w = \frac{\dot{\tau}\dot{\hat{\tilde{\tau}}}}{\tilde{\tau}\hat{\tau}}, \tag{8}$$

and the variable v introduced above. We have $u = -P_{31} = Q_{31}$, $z = P_{32} = -Q_{32}$, and $w = R_{33}$. The third conservation law becomes

$$a_1(\hat{u} - \tilde{u}) + a_2(\tilde{z} - \hat{z}) + a_3(w - \dot{w}) + a_4(v - \dot{v}) = 0. \tag{9}$$

Taking logarithms we find

$$\ln(u) = \frac{(\tilde{S} - 1)(\tilde{S} - \dot{S})}{\tilde{S}}c, \quad \ln(z) = \frac{(\hat{S} - 1)(\hat{S} - \dot{S})}{\hat{S}}c, \quad \ln(w) = \frac{(\tilde{S} - \dot{S})(\hat{S} - \tilde{S})}{\dot{S}}c.$$

One can now derive quotient equations which are either ratios of quadratic terms

$$\frac{\hat{\hat{\tilde{u}}}\hat{\tilde{u}}}{\dot{\hat{\hat{\tilde{u}}}}\hat{\tilde{u}}} = \frac{\hat{\tilde{\tilde{z}}}\dot{\hat{z}}}{\dot{\hat{\tilde{\tilde{z}}}}\hat{\tilde{\tilde{z}}}} = \frac{\dot{\hat{\tilde{w}}}\dot{w}}{\dot{\tilde{w}}\dot{\hat{w}}} = \frac{\hat{\tilde{v}}\dot{\dot{v}}}{\dot{\tilde{v}}\dot{\hat{v}}},$$

or ratios of linear terms

$$\frac{\tilde{\hat{u}}}{\tilde{u}} = \frac{\tilde{v}}{\dot{v}}, \quad \frac{\tilde{\hat{u}}}{\dot{\tilde{u}}} = \frac{\dot{\tilde{w}}}{\dot{w}}, \quad \frac{\tilde{\hat{z}}}{\hat{z}} = \frac{\hat{v}}{\dot{v}}, \quad \frac{\tilde{\hat{z}}}{\dot{\hat{z}}} = \frac{\dot{\hat{w}}}{\dot{w}}, \tag{10}$$

of which only three are independent. In the sequel, we will refer to the system of quotient and difference equations (9) and (10) as the Q^{3}D-system.

4. The *N*-soliton solution

4.1. 1-soliton

Equation (1) admits the 1-soliton solution $\tau_{k,l,m} = 1 + c_1 x_1^k y_1^l z_1^m$ with dispersion relation $Q_1 = 0$, where

$$
\begin{aligned}
Q_i =& y_i z_i (x_i - 1)(x_i - y_i)(x_i - z_i) a_1 + x_i z_i (y_i - 1)(y_i - x_i)(y_i - z_i) a_2 \\
&+ x_i y_i (z_i - 1)(z_i - x_i)(z_i - y_i) a_3 + x_i y_i z_i (x_i - 1)(y_i - 1)(z_i - 1) a_4.
\end{aligned} \tag{11}
$$

In the sequel we will use the following notation, $x_{ij} = x_i x_j$, $c_{ij} = c_i c_j$, and if $Q_i = Q(x_i, y_i, z_i)$ then $Q_{ij} = Q(x_{ij}, y_{ij}, z_{ij})$.

4.2. 2-soliton

Equation (1) admits the 2-soliton solution

$$\tau_{k,l,m} = 1 + c_1 x_1^k y_1^l z_1^m + c_2 x_2^k y_2^l z_2^m + c_1 c_2 R_{12} x_{12}^k y_{12}^l z_{12}^m$$

where $Q_1 = Q_2 = 0$,

$$R_{ij} = \frac{a_1 S_1^{ij} + a_2 S_2^{ij} + a_3 S_3^{ij} + a_4 S_4^{ij}}{Q_{ij}},$$

with

$$
\begin{aligned}
S_1^{ij} =& \Big((x_i - x_j)(x_j y_i - y_j x_i)(x_{ij} - z_{ij}) + (x_i - x_j)(x_j z_i - z_j x_i)(x_{ij} - y_{ij}) \\
&+ (x_j z_i - z_j x_i)(x_j y_i - y_j x_i)(1 - x_{ij}) \Big) y_{ij} z_{ij}, \\
S_4^{ij} =& \Big((1 - x_{ij})(y_i - y_j)(z_i - z_j) + (x_i - x_j)(1 - y_{ij})(z_i - z_j) \\
&+ (x_i - x_j)(y_i - y_j)(1 - z_{ij}) \Big) x_{ij} y_{ij} z_{ij},
\end{aligned}
$$

and S_k^{ij} for $k = 2$, resp. $k = 3$, are obtained from S_1^{ij} by interchanging the symbols x and y, respectively x and z. This has been checked by direct computation, using a Groebner basis in Maple [25].

4.3. N-soliton

Let $P(N)$ denote the powerset of the string $12\ldots N$, e.g. we write

$$P(3) = \{\varepsilon, 1, 2, 3, 12, 23, 13, 123\},$$

where ε is the empty string, and let $P_2(S)$ be the subset of the powerset of a string S containing all 2-letter substrings, e.g.

$$P_2(123) = \{12, 23, 13\}.$$

Conjecture 1. Equation (1) admits the following N-soliton solution:

$$\tau_{k,l,m} = \sum_{w \in P(N)} \Big(\prod_{v \in P_2(w)} R_v \Big) c_w x_w^k y_w^l z_w^m, \qquad \text{with } Q_i = 0,\ i \in \{1, 2, \ldots, N\}.$$

Note that in the above formula $c_\varepsilon = x_\varepsilon = \cdots = 1$ is understood. The formula can be computationally checked as follows: taking particular values for a_1, a_2, a_3 and a_4, one can find rational points $p_i = (x_i, y_i, z_i) \in \mathbb{Q}^3$ such that $Q_i = 0$. Using $N \in \mathbb{N}$ points, one substitutes the N-soliton solution, which contains N arbitrary constants $c_1, \ldots, c_N$, into the equation for fixed points $(k, l, m) \in \mathbb{Z}^3$. For example, taking $(a_1, a_2, a_3, a_4) = (1, 2, 3, 2)$ the following points

$$p_1 = (2, 4, 2/3), \qquad p_2 = (6, 21, -14), \qquad p_3 = (7, 14, -6),$$
$$p_4 = (8, 15, -40/9), \qquad p_5 = (14, 80, -560), \qquad p_6 = (18, 120, -15/2)$$

satisfy $Q_i = 0$. Taking $k = -2, l = 1, m = 3$ one needs to verify that

$$\begin{aligned}&\tau_{-3,2,4}\tau_{-1,1,3}\tau_{-1,1,4}\tau_{-1,2,3} + 2\,\tau_{-2,1,3}\tau_{-2,2,4}\tau_{-1,1,4}\tau_{-1,2,3} - \tau_{-2,1,4}\tau_{-2,2,3}\tau_{-2,2,4}\tau_{0,1,3}\\ &- 2\,\tau_{-2,1,4}\tau_{-2,2,3}\tau_{-1,1,3}\tau_{-1,2,4} + 3\,\tau_{-2,1,4}\tau_{-2,2,4}\tau_{-1,1,4}\tau_{-1,2,2} - 2\,\tau_{-2,1,4}\tau_{-2,3,3}\tau_{-1,1,3}\tau_{-1,1,4}\\ &- 3\,\tau_{-2,1,5}\tau_{-2,2,3}\tau_{-1,1,3}\tau_{-1,2,3} + 2\,\tau_{-2,2,3}\tau_{-2,2,4}\tau_{-1,0,4}\tau_{-1,2,3} \qquad (12)\end{aligned}$$

vanishes. Using the above 6 points p_i the value of the 6-soliton solution at $(k, l, m) = (-3, 2, 4)$ is

$$\begin{aligned}\tau_{-3,2,4} = 1 &+ \frac{32\,c_1}{81} + \frac{235\,298\,c_2}{3} + \frac{5184\,c_3}{7} + \frac{125\,000\,c_4}{729} - \frac{9973\,408\,256.10^8\,c_1c_2c_3c_4}{2850\,829\,229\,061}\\
&+ \frac{17\,537\,436\,614\,656.10^{14}\,c_1c_4c_5}{304\,882\,184\,692\,881} - \frac{286\,643\,773\,308\,928.10^{13}\,c_2c_4c_5}{5379\,614\,362\,287}\\
&- \frac{31\,023\,435\,087\,872.10^{14}\,c_3c_4c_5}{1827\,893\,357\,279\,451} - \frac{8355\,684\,882\,055\,168.10^7\,c_1c_2c_5}{3557\,331} - \frac{244.10^6\,c_1c_4}{2735\,937}\\
&+ \frac{419\,082\,155\,327\,488.10^{10}\,c_1c_3c_5}{79\,592\,065\,203} - \frac{25\,192\,657\,019\,901\,837\,312.10^7\,c_2c_3c_5}{414\,577\,637\,413}\\
&+ \frac{15\,625\,c_6}{2} + 229\,376.10^6\,c_5 + \frac{430\,515.10^5\,c_3c_6}{31\,3747} + \frac{1220\,703\,125\,c_4c_6}{34\,992}\\
&- \frac{6272.10^{13}\,c_5c_6}{339} + \frac{5.10^5\,c_1c_6}{1539} - \frac{1838\,265\,625\,c_2c_6}{422}\\
&+ \frac{4563\,788\,408\,614\,224\,681\,500\,672.10^{17}\,c_1c_2c_3c_4c_5}{3335\,745\,327\,609\,453\,768\,757\,318\,923} - \frac{1075\,648.10^6\,c_2c_3c_4}{253\,204\,479}\\
&+ \frac{9415\,684\,768.10^6\,c_1c_2c_4}{1595\,051\,271} - \frac{74\,176.10^8\,c_1c_3c_4}{10\,556\,764\,911} + \frac{359\,696\,691\,200\,c_1c_2c_3}{13\,161}\\
&- \frac{323\,830\,284\,288.10^6\,c_2c_5}{367} + \frac{68\,956\,750\,243\,187\,158\,016.10^{25}\,c_1c_2c_3c_4c_5c_6}{13\,520\,106\,051\,588\,549\,460\,696\,520\,569\,492\,377}\\
&+ \frac{203\,190\,312\,540\,790\,063\,104.10^{17}\,c_1c_2c_3c_5c_6}{22\,721\,904\,043\,520\,272\,815\,643} - \frac{270\,945\,647.10^{21}\,c_2c_4c_5c_6}{3405\,295\,891\,327\,671}\\
&+ \frac{2558\,749\,998\,443\,072.10^{22}\,c_1c_2c_4c_5c_6}{87\,172\,894\,850\,121\,902\,722\,923} + \frac{512\,857\,367\,787\,136.10^{25}\,c_1c_3c_4c_5c_6}{17\,456\,412\,275\,319\,361\,592\,913\,150\,717}\\
&- \frac{5065\,859\,375.10^8\,c_1c_3c_4c_6}{1419\,494\,280\,227\,793} - \frac{8.10^6\,c_3c_4}{22\,869} + \frac{1197\,498\,441\,289\,024.10^{22}\,c_2c_3c_4c_5c_6}{3344\,920\,733\,568\,156\,174\,088\,032\,717}\\
&+ \frac{486\,525\,134\,375.10^{10}\,c_1c_2c_3c_4c_6}{3851\,564\,365\,825\,970\,013} + \frac{433\,061\,888.10^7\,c_1c_5}{29\,079} - \frac{5190\,429\,687\,500\,c_3c_4c_6}{3075\,034\,347}\end{aligned}$$

$$
\begin{aligned}
&-\frac{3785\,197\,879\,296.10^{7}\,c_3c_5}{1802\,479}-\frac{152\,276\,992.10^{13}\,c_4c_5}{226\,286\,703}-\frac{23\,683\,072.10^{14}\,c_1c_5c_6}{187\,297\,839}\\
&+\frac{1353\,499\,793\,462\,147\,416\,064.10^{14}\,c_1c_2c_4c_5}{7248\,099\,679\,897\,056\,849}-\frac{24\,118\,045.10^{5}\,c_1c_2c_6}{324\,729}\\
&+\frac{34\,110\,994\,995.10^{5}\,c_2c_3c_6}{5926\,981\,771}-\frac{297\,851\,562\,500\,c_1c_4c_6}{155\,948\,409}+\frac{287\,875.10^{18}\,c_4c_5c_6}{2036\,580\,327}\\
&+\frac{308\,710\,976\,c_1c_2}{243}+\frac{143\,614\,501\,953\,125\,c_2c_4c_6}{358\,705\,908}-\frac{8504.10^{8}\,c_1c_3c_6}{38\,590\,881}\\
&-\frac{4427\,367\,168.10^{13}\,c_2c_5c_6}{8750\,381}-\frac{972\,800\,c_1c_3}{861}-\frac{224\,599\,144.10^{8}\,c_1c_2c_3c_6}{5926\,981\,771}\\
&-\frac{1916\,804\,736\,c_2c_3}{4387}-\frac{117\,649.10^{6}\,c_2c_4}{425\,007}+\frac{929\,743\,819\,565\,759\,987\,712.10^{10}\,c_1c_2c_3c_5}{148\,833\,371\,831\,267}\\
&+\frac{5239\,147\,585\,536.10^{14}\,c_3c_5c_6}{1304\,163\,853\,181}-\frac{16\,971\,308\,117\,950\,201\,856.10^{17}\,c_1c_3c_4c_5}{302\,920\,719\,026\,139\,857\,929\,671}\\
&+\frac{8342\,573\,976\,349\,835\,264.10^{14}\,c_2c_3c_4c_5}{825\,274\,865\,866\,379\,324\,583}-\frac{183\,176\,122\,990\,592.10^{17}\,c_1c_3c_5c_6}{172\,763\,889\,794\,740\,251}\\
&-\frac{66\,307\,976.10^{22}\,c_1c_4c_5c_6}{52\,134\,853\,582\,482\,651}+\frac{4984\,888\,671\,875.10^{4}\,c_2c_3c_4c_6}{342\,087\,606\,876\,807}\\
&-\frac{52\,304\,285\,961\,241\,509\,888.10^{14}\,c_2c_3c_5c_6}{63\,292\,211\,820\,390\,732\,077}+\frac{8906\,247\,704.10^{22}\,c_3c_4c_5c_6}{105\,336\,010\,499\,942\,922\,777}\\
&-\frac{574\,687\,791\,015\,625.10^{4}\,c_1c_2c_4c_6}{6394\,560\,545\,439}-\frac{228\,475\,758\,493\,696.10^{14}\,c_1c_2c_5c_6}{1611\,531\,417\,627},
\end{aligned}
$$

and we obtain similar expressions for the values of the 6-soliton solution at the other 13 lattice points of the stencil, see figure 1. Substituting these expressions into (12) gives zero. This has been checked also for other values of a_i, other points p_j and other values for k, l, m.

We have also performed another computational verification, this time of the 3-soliton solution. Starting with expressions for p_1, p_2, p_3 of the form $p_i = b_i x + c_i$ where $b_i, c_i \in \mathbb{Q}$ are randomly chosen and x is a parameter, we have solved the linear system $Q_{12} = Q_{13} = Q_{23} = 0$ for a_1, a_2, a_3, a_4, and verified the solution for a range of values for k, l, m.

In figure 2 we have plotted two cross sections of a three soliton solution.

5. Laurent property

Consider an ordinary difference equation of order d,

$$\tau_n = \frac{P(\tau_{n-d}, \ldots, \tau_{n-1})}{Q(\tau_{n-d}, \ldots, \tau_{n-1})}, \tag{13}$$

where P is a polynomial and Q is a monomial. Let $\mathcal{R}$ be the ring of coefficients. From a set of d initial values $U = \{\tau_k\}_{0 \leqslant k < d}$, one finds τ_n as rational functions of the initial values, given by

$$\tau_n = \frac{p_n(\tau_0, \ldots, \tau_{d-1})}{q_n(\tau_0, \ldots, \tau_{d-1})}, \tag{14}$$

with greatest common divisor $\gcd(p_n, q_n) = 1$. By definition, if $q_n \in \mathcal{R}[U]$ is a monomial for all $n \geqslant 0$, then (13) has the Laurent property. The first examples of recurrences with the Laurent property were discovered by Michael Somos in the 1980s [12]. Since then many more have been found [2, 6, 8, 15, 24], and the Laurent property is a central feature of cluster algebras [9, 10]. In [27, definition 2.11] the author defines the Laurent property for discrete bilinear equations. The idea is that a lattice equation has the Laurent property if all *good* initial value problems have the Laurent property. The author points out that not all well-posed, see [34], initial value problems are good. Certainly, the initial value problems obtained from (doubly periodic) reductions given below, see (15), are good.

In [14] a more specific Laurent property was introduced, where the terms are Laurent polynomials in some of the variables but polynomial in others. The form of (13) guarantees that all components q_n are monomials for $0 \leqslant n \leqslant d$. Suppose these monomials depend on a subset of the initial values $V \subset U$. The following conditions guarantee that q_n is a monomial $\in \mathcal{R}[V]$ for all $n \geqslant 0$, see [14, theorem 2].

Theorem 2. *Suppose that q_d is a monomial in $\mathcal{R}[V]$. If p_d is coprime to p_{d+k} for all $k = 1, \dots, d$, and $q_m \in \mathcal{R}[V]$ is a monomial for $d+1 \leqslant m \leqslant 2d$, then (13) has the following Laurent property: all iterates are Laurent polynomials in the variables from V and they are polynomial in the remaining variables from $W = U \setminus V$.*

Introducing the variable $n = z_1 k + z_2 l + z_3 m$, where we take z_1, z_2, z_3 to be non-negative integers such that $\gcd(z_1, z_2, z_3) = 1$, and performing a reduction $\tau_{k,l,m} \to \tau_n$, one obtains the ordinary difference equation

$$\begin{aligned} 0 = {} & a_1 \left(\tau_{n+z_1}\tau_{n+z_1+z_2}\tau_{n+z_1+z_3}\tau_{n-z_1+z_2+z_3} - \tau_{n+2z_1}\tau_{n+z_2}\tau_{n+z_3}\tau_{n+z_2+z_3}\right) \\ & + a_2 \left(\tau_{n+z_2}\tau_{n+z_1+z_2}\tau_{n+z_2+z_3}\tau_{n+z_1-z_2+z_3} - \tau_{n+z_1}\tau_{n+2z_2}\tau_{n+z_3}\tau_{n+z_1+z_3}\right) \\ & + a_3 \left(\tau_{n+z_3}\tau_{n+z_1+z_3}\tau_{n+z_2+z_3}\tau_{n+z_1+z_2-z_3} - \tau_{n+z_1}\tau_{n+z_2}\tau_{n+2z_3}\tau_{n+z_1+z_2}\right) \\ & + a_4 \left(\tau_n\tau_{n+z_1+z_2}\tau_{n+z_1+z_3}\tau_{n+z_2+z_3} - \tau_{n+z_1}\tau_{n+z_2}\tau_{n+z_3}\tau_{n+z_1+z_2+z_3}\right), \end{aligned} \tag{15}$$

which has order

$$d = \max(2z_1, 2z_2, 2z_2, z_1 + z_2 + z_3) - \min(0, z_1 + z_2 - z_3, z_1 + z_3 - z_2, z_2 + z_3 - z_1).$$

Conjecture 3. The iterates τ_n are Laurent polynomials in the initial values τ_i, with $i = p, p+1, \dots, d-p-1$ where

$$p = \min(z_1, z_2, z_2) - \min(0, z_1 + z_2 - z_3, z_1 + z_3 - z_2, z_2 + z_3 - z_1),$$

and polynomial in the others, $\tau_0, \tau_1, \dots, \tau_{p-1}, \tau_{d-p}, \dots, \tau_{d-2}, \tau_{d-1}$.

This conjecture has been proven, using Theorem 2 and [25], for $z_1 = z_2 = 1, 1 \leqslant z_3 \leqslant 20$, for $z_1 = 1, z_2 = 2, z_3 = 3$, and some but not all of the conditions of Theorem 2 have been verified for all co-prime $z_1 < z_2 < z_3 \leqslant 10$.

6. Degree growth

Given an ordinary difference equation of the form (13) one can define an integer sequence $\{d_n^p\}_{n=0}^{\infty}$ where d_n^p denotes the degree of the polynomial p_n defined by (14). According to the *degree growth conjecture* [7, 17] we have

- Growth is linear in $n \Longrightarrow$ equation is linearizable.
- Growth is polynomial in $n \Longrightarrow$ equation is integrable.

- Growth is exponential in $n \Longrightarrow$ equation is non-integrable.

Conjecture 4. For all positive integers z_1, z_2, z_3 such that $\gcd(z_1, z_2, z_3) = 1$ equation (15) has quadratic growth.

We have verified the following. Choosing (randomly) rational values for the coefficients a_i, starting with rational initial values $\tau_0, \ldots, \tau_{d-2}$ and letting $\tau_{d-1} = a + bx$, where a, b are rational values and x a parameter, we have calculated up to a 150 iterates until the degree (in x) exceeded 250. Taking the second difference of the degree sequence yielded a periodic sequence in almost all cases with $1 \leqslant z_1 \leqslant 4$, $1 \leqslant z_2 \leqslant z_3 \leqslant 7$. In two cases more iterations were required. Keeping the maximal degree fixed at 500, for $z = (1, 1, 7)$ we calculated 370 iterations and found that the period of the second difference is 259, for $z = (3, 7, 7)$ we calculated 354 iterations and found that the period of the second difference is 240. Curiously, the leading order terms are all of the form $(M^{z_1}_{z_2,z_3})^{-1} n^2$ with

$$M^1 = \begin{bmatrix} 2 & 4 & 15 & 40 & 85 & 156 & 259 \\ 4 & 7 & 12 & 25 & 60 & 94 & 172 \\ 15 & 12 & 16 & 24 & 40 & 76 & 150 \\ 40 & 25 & 24 & 29 & 40 & 60 & 108 \\ 85 & 60 & 40 & 40 & 46 & 60 & 82 \\ 156 & 94 & 76 & 60 & 60 & 67 & 84 \\ 259 & 172 & 150 & 108 & 82 & 84 & 92 \end{bmatrix}, \quad M^2 = \begin{bmatrix} 4 & 7 & 12 & 25 & 60 & 94 & 172 \\ 7 & x & 15 & x & 40 & x & 154 \\ 12 & 15 & 28 & 25 & 60 & 55 & 132 \\ 25 & x & 25 & x & 40 & x & 76 \\ 60 & 40 & 60 & 40 & 84 & 60 & 140 \\ 94 & x & 55 & x & 60 & x & 82 \\ 172 & 154 & 132 & 76 & 140 & 82 & 172 \end{bmatrix},$$

$$M^3 = \begin{bmatrix} 15 & 12 & 16 & 24 & 40 & 76 & 150 \\ 12 & 15 & 28 & 25 & 60 & 55 & 132 \\ 16 & 28 & x & 40 & 40 & x & 77 \\ 24 & 25 & 40 & 69 & 60 & 55 & 168 \\ 40 & 60 & 40 & 60 & 114 & 76 & 76 \\ 76 & 55 & x & 55 & 76 & x & 108 \\ 150 & 132 & 77 & 168 & 76 & 108 & 240 \end{bmatrix}, \quad M^4 = \begin{bmatrix} 40 & 25 & 24 & 29 & 40 & 60 & 108 \\ 25 & x & 25 & x & 40 & x & 76 \\ 24 & 25 & 40 & 69 & 60 & 55 & 168 \\ 29 & x & 69 & x & 85 & x & 77 \\ 40 & 40 & 60 & 85 & 136 & 94 & 132 \\ 60 & x & 55 & x & 94 & x & 150 \\ 108 & 76 & 168 & 77 & 132 & 150 & 296 \end{bmatrix},$$

where x indicates that $\gcd(z_1, z_2, z_3) > 1$.

7. Reductions to 2D integrable lattice equations

We give some reductions to integrable 2D lattice equations known in the literature.

- Setting $\dot{} = \hat{}$, $u = e$, $v = \tilde{q}$, $z = w = 0$ and $a_1 + a_4 = 0$ the Q^{3}D-system reduces to Rutishauser's QD-algorithm

$$\tilde{e} + \tilde{q} = \hat{e} + \tilde{\tilde{q}}, \qquad \hat{e}\hat{q} = e\tilde{q}.$$

- Taking $z = a_2 = 0$ and $a_1 = a_3 = -a_4 = 1$ and introducing variables $i = k - l$, $j = 3l + m$, $\tau_{k,l,m} = \Delta_i^j$, equation (1) reduces to the higher analogue of the discrete-time Toda (HADT) equation [32, equation (3.18)],

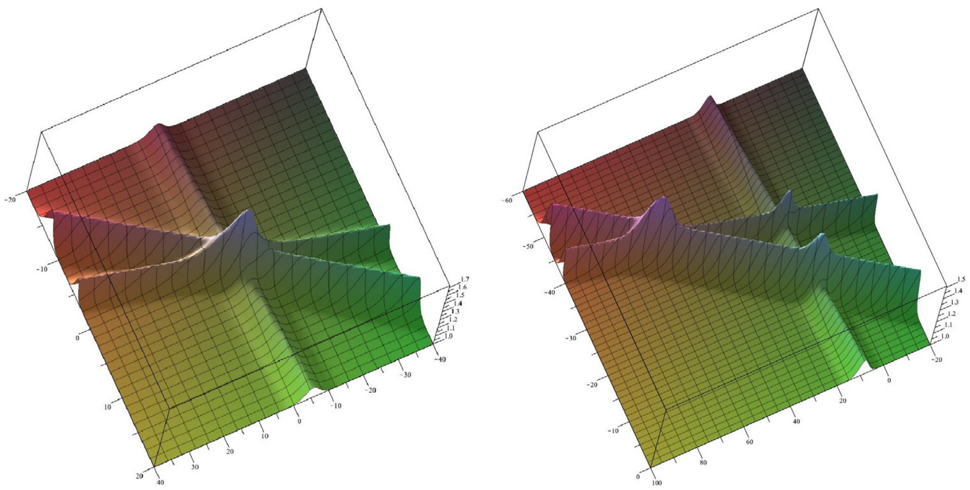

Figure 2. Two cross sections, $m = 0$ resp. $m = 50$, of the function u defined in (8) where τ is the three soliton solution of dual AKP with $(a_1, a_2, a_3, a_4) = (1, 2, 3, 2)$ and $p_1 = (\frac{1}{5}, \frac{12}{13}, \frac{18}{65})$, $p_2 = (\frac{11}{31}, \frac{15}{32}, \frac{495}{496})$, $p_3 = (2, 4, \frac{2}{3})$, and $c_1 = c_2 = c_3 = 1$.

$$\begin{aligned}
&\Delta^{j}_{i+1}\left(\Delta^{j+4}_{i-2}\Delta^{j+1}_{i+1}\Delta^{j+3}_{i} - \Delta^{j+2}_{i}\Delta^{j+3}_{i-1}\Delta^{j+3}_{i} + \Delta^{j+3}_{i-1}\Delta^{j+1}_{i}\Delta^{j+4}_{i}\right)\\
&\quad = \Delta^{j+4}_{i-1}\left(-\Delta^{j}_{i+2}\Delta^{j+1}_{i}\Delta^{j+3}_{i-1} + \Delta^{j+2}_{i}\Delta^{j+1}_{i}\Delta^{j+1}_{i+1} - \Delta^{j}_{i}\Delta^{j+1}_{i+1}\Delta^{j+3}_{i}\right),
\end{aligned}$$

and the Q^3D-system reduces to the QQD-system [32, equation (1.4)],

$$\begin{aligned}
u_{i,3+j} + v_{i+1,j+1} + w_{i+1,j} &= u_{i+2,j} + v_{i+1,j} + w_{i+1,j+1}\\
u_{i,3+j}v_{i,j+1} &= u_{i+1,j}v_{i+1,j}\\
u_{i,3+j}w_{i,j+1} &= u_{i+1,j+1}w_{i+1,j+1}.
\end{aligned}$$

- By introducing some special bi-orthogonal polynomials, in [3] the so-called discrete hungry quotient-difference (dhQD) algorithm and a system related to the QD-type discrete hungry Lotka–Volterra (QD-type dhLV) system have been derived, as well as hungry forms of the HADT-equation (hHADT) and the QQD scheme (hQQD). These systems are all reductions of the Q^3D system, or of the dual to the AKP equation (1).

  Setting $z = w = 0$, $\tilde{u} = q$, $v = e$ and introducing $i = k$, $j = pl + m$ we get QD-type dhLV [3, equations (6) and (7)],

$$\begin{aligned}
e_{i,j} + q_{i,j} &= e_{i,j+1} + q_{i-1,j+p}\\
e_{i,j+1}q_{i,j+p} &= e_{i+1,j}q_{i,j}.
\end{aligned}$$

  Setting $z = w = 0$, $\tilde{u} = q$, $v = e$ and introducing $i = k$, $j = l + pm$ we get dhQD [32, equations (9) and (10)],

$$\begin{aligned}
e_{i,j} + q_{i,j} &= e_{i,j+p} + q_{i-1,j+1}\\
e_{i,j+p}q_{i,j+1} &= e_{i+1,j}q_{i,j}.
\end{aligned}$$

With $z = 0$ the reduction $i = k - l$, $j = (p+2)l + pm$ yield hQQD [3, equation (23)],

$$\begin{aligned} u_{i,j+p+2} + v_{i+1,j+p} + w_{i+1,j} &= u_{i+2,j} + v_{i+1,j} + w_{i+1,j+p} \\ u_{i,j+p+2} v_{i,j+p} &= u_{i+1,j} v_{i+1,j} \\ u_{i,j+2} w_{i,j} &= u_{i+1,j} w_{i+1,j}. \end{aligned}$$

Performing the same reduction on equation (1), with $a_2 = 0$, gives the hHADT equation [3, equation (18)],

$$\begin{aligned} &\left(\Delta_{i-2}^{j+2p+2}\Delta_{i+1}^{j+p}\Delta_i^{j+p+2} - \Delta_i^{j+2p}\Delta_{i-1}^{j+p+2}\Delta_i^{j+p+2} + \Delta_i^{j+2p+2}\Delta_i^{j+p}\Delta_{i-1}^{j+p+2}\right)\Delta_{i+1}^{j} \\ &= \left(\Delta_{i+2}^{j}\Delta_i^{j+p}\Delta_{i-1}^{j+p+2} - \Delta_i^{j+2}\Delta_i^{j+p}\Delta_{i+1}^{j+p} + \Delta_i^{j}\Delta_{i+1}^{j+p}\Delta_i^{j+p+2}\right)\Delta_{i-1}^{j+2p+2}. \end{aligned}$$

8. Conclusion

In this paper we have generalized the concept of duality introduced in [29] for ordinary difference equations (OΔEs) to the realm of lattice equations (PΔEs). The dAKP equation (1) and the AKP equation (2) are dual to each other. Generally speaking, dual equations to integrable equations do not need to be integrable themselves; the only thing that is guaranteed is the existence of integrals (for OΔEs), or conservation laws (for PΔEs). However, our equation (2) unifies a number of known (hierarchies of) integrable 2D lattice equations, which arise as reductions. Together with the support we have provided for our conjectures, that equation (1) admits an N-soliton solution, and its reductions have the Laurent property and zero algebraic entropy, we believe it is a new *integrable* 3D lattice equation.

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