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Closed-form expressions for integrals of MKdV and sine-Gordon maps

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Abstract
We present closed-form expressions for approximately N integrals of $2N$-dimensional maps. The maps are obtained by travelling wave reductions of the modified Korteweg–de Vries equation and of the sine-Gordon equation, respectively. We provide the integrating factors corresponding to the integrals. Moreover we show how the integrals and the integrating factors relate to the staircase method.

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1. Introduction

In recent years, the study of discrete integrable systems has witnessed a revolutionary phase of expansion. Examples of discrete nonlinear equations that are in a certain sense solvable were discovered and the first instances of classification of such systems have appeared [1, 4]. They have applications, e.g. in statistical mechanics, quantum gravity, classical mechanics, solid state physics and in the convergence of algorithms, cf [2, 3, 9, 13].

Two main classes of discrete integrable systems that may be distinguished are integrable partial difference equations (P∆E), and integrable ordinary difference equations (O∆E). The latter are equivalent to integrable mappings. A connection between the two classes is that many integrable maps can be obtained from integrable P∆E's by imposing periodic boundary conditions [6, 7]. In this paper we study (the integrability of) two families of maps obtained in this way from the discrete modified Korteweg–de Vries (MKdV) P∆E and from the sine-Gordon P∆E, respectively. We present for the first time simple and elegant closed-form expressions for (almost) all their first integrals.

The families of ordinary difference equations, labelled by one integer $p \in \mathbb{N}$, are the modified Korteweg–de Vries O∆E

$$f_n := \alpha_1(v_n v_{n+p} - v_{n+1} v_{n+p+1}) + \alpha_2 v_n v_{n+1} - \alpha_3 v_{n+p} v_{n+p+1} = 0, \quad (1)$$

and the sine-Gordon OΔE

$$\tilde{f}_n := \beta_1(v_n v_{n+p+1} - v_{n+1}v_{n+p}) + \beta_2 v_n v_{n+1} v_{n+p} v_{n+p+1} - \beta_3 = 0. \tag{2}$$

In these equations $\alpha_i, \beta_i, v_i \in \mathbb{R}$, the subscripts on v denote the values of the independent variable. In general we will denote quantities related to the sine-Gordon equation by a superscript wiggle. Both equations is equivalent to a mapping $\mathbb{R}^{p+1} \to \mathbb{R}^{p+1}$:

$$(v_0, v_1, \ldots, v_p) \mapsto (v_1, v_2, \ldots, v_p, v_{p+1}),$$

where we have set $n = 0$ and v_{p+1} is given by the solution of either equation (1) or (2):

$$v_{p+1} = v_0 \frac{\alpha_1 v_p + \alpha_2 v_1}{\alpha_1 v_1 + \alpha_3 v_p}, \qquad v_{p+1} = v_0^{-1} \frac{\beta_1 v_1 v_p + \beta_3}{\beta_2 v_1 v_p + \beta_1},$$

respectively. We denote the shift operator by S, e.g. we write $S^j(v_i) = v_{i+j}$. It is plain that a function $I(v_0, v_1, \ldots, v_p)$ of $p+1$ variables is an invariant of the mapping obtained from an equation $f(v_0, v_1, \ldots, v_{p+1}) = 0$ if there is a $\Lambda(v_0, v_1, \ldots, v_p)$ such that $S(I) - I = f\Lambda$. The functions I and Λ are called an *integral*, and an *integrating factor* of the equation, or of the mapping, respectively.

In this paper we give closed-form expressions for $\lfloor p/2 \rfloor$ integrals of the MKdV mapping, and $\lceil p/2 \rceil$ integrals of the sine-Gordon mapping. For any even positive q less than $p+1$,

$$I_q^p = \sum_{0 \leqslant c_1 < c_2 < \cdots < c_{q-1} < p} \alpha_1 \left(v_0 v_p \prod_{i=1}^{q-1} (v_{c_i} v_{c_i+1})^{(-1)^i} + \frac{1}{v_0 v_p} \prod_{i=1}^{q-1} (v_{c_i} v_{c_i+1})^{(-1)^{i+1}} \right)$$
$$+ \sum_{0 \leqslant c_1 < c_2 < \cdots < c_q < p} \left(\alpha_2 \prod_{i=1}^{q} (v_{c_i} v_{c_i+1})^{(-1)^{i+1}} + \alpha_3 \prod_{i=1}^{q} (v_{c_i} v_{c_i+1})^{(-1)^i} \right)$$

is a non-trivial integral of the MKdV mapping. We will show (theorem 1) that $S\big(I_q^p\big) - I_q^p = f_0 \Lambda_q^p$, with

$$\Lambda_q^p = \sum_{0 < c_1 < c_2 < \cdots < c_{q-1} < p} \left(\frac{1}{v_0 v_1 v_p v_{p+1}} \prod_{i=1}^{q-1} (v_{c_i} v_{c_i+1})^{(-1)^{i+1}} - \prod_{i=1}^{q-1} (v_{c_i} v_{c_i+1})^{(-1)^i} \right).$$

These integrals are not new in the sense that an algorithm was previously found by which they can be obtained (although *not* in this elegant and succinct closed form). This is the staircase method, which we briefly describe in section 4. We will actually prove that the above integrals, and the integrating factors, are exactly those obtainable by the staircase method.

The sine-Gordon equation has a similar structure as the MKdV equation and it gives similar results. These results will be stated without providing the detailed calculations. The structure of their proofs is the same as for MKdV. For any even non-negative q smaller than p,

$$\tilde{I}_q^p = \sum_{0 \leqslant c_1 < c_2 < \cdots < c_q < p} \beta_1 \left(\frac{v_p}{v_0} \prod_{i=1}^{q} (v_{c_i} v_{c_i+1})^{(-1)^{i+1}} + \frac{v_0}{v_p} \prod_{i=1}^{q} (v_{c_i} v_{c_i+1})^{(-1)^i} \right)$$
$$+ \sum_{0 \leqslant c_1 < c_2 < \cdots < c_{q+1} < p} \left(\beta_2 \prod_{i=1}^{q+1} (v_{c_i} v_{c_i+1})^{(-1)^{i+1}} + \beta_3 \prod_{i=1}^{q+1} (v_{c_i} v_{c_i+1})^{(-1)^i} \right)$$

is a non-trivial integral of the sine-Gordon mapping. We have $S\big(\tilde{I}_q^p\big) - \tilde{I}_q^p = \tilde{f}_0 \tilde{\Lambda}_q^p$, with

$$\tilde{\Lambda}_q^p = \sum_{0 < c_1 < c_2 < \cdots < c_{q-1} < p} \left(\frac{1}{v_0 v_1} \prod_{i=1}^{q} (v_{c_i} v_{c_i+1})^{(-1)^{i+1}} - \frac{1}{v_p v_{p+1}} \prod_{i=1}^{q} (v_{c_i} v_{c_i+1})^{(-1)^i} \right).$$

2. A nice combinatorial object: a multi-sum of products

The following involutive transformation will play a role, $\tau : v_j \mapsto v_j^{-1}$. The image under τ of a function f of variables v_j will be denoted f^τ. In fact both the MKdV and the sine-Gordon equations are invariant under the composition of τ and interchanging the parameters $\alpha_2 \leftrightarrow \alpha_3$ and $\beta_2 \leftrightarrow \beta_3$, respectively.

Using the following notation, the above integrals and integrating factors can be expressed quite conveniently. We define, with $a, b, n, \epsilon \in \mathbb{Z}$,

$$\Theta_{n,\epsilon}^{a,b} = \sum_{a \leqslant i_1 < i_2 < \cdots < i_n \leqslant b} \prod_{j=1}^{n} (v_{i_j} v_{i_j+1})^{(-1)^{j+\epsilon}}.$$

From the definitions we obtain

- $\Theta_{n,\epsilon+1}^{a,b} = \Theta_{n,\epsilon-1}^{a,b}$,
- $\tau\left(\Theta_{n,\epsilon}^{a,b}\right) = \Theta_{n,\epsilon+1}^{a,b}$,
- $S\left(\Theta_{n,\epsilon}^{a,b}\right) = \Theta_{n,\epsilon}^{a+1,b+1}$,
- $\Theta_{0,\epsilon}^{a,b} = 1$,
- $\Theta_{n,\epsilon}^{a,b} = 0$ when either $n > \max(0, b-a+1)$ or $n < 0$.

Moreover we have the identities

$$\Theta_{n,\epsilon}^{a,b} = \Theta_{n,\epsilon}^{a+1,b} + (v_a v_{a+1})^{(-1)^{1+\epsilon}} \Theta_{n-1,\epsilon\pm1}^{a+1,b} \tag{3}$$

and

$$\Theta_{n,\epsilon}^{a,b} = \Theta_{n,\epsilon}^{a,b-1} + (v_b v_{b+1})^{(-1)^{n+\epsilon}} \Theta_{n-1,\epsilon}^{a,b-1}. \tag{4}$$

Our Θ is a special multi-sum of products, instead of $F_n = v_n v_{n+1}$ one could take any function F of one variable. Also the latter identities are special cases of more general identities where Θ's are expressed as sums of products of Θ's. However, the above definition and identities suffice for our purpose.

3. The integrals

In this section we prove that the integrals I_q^p given in the introduction are invariants of the MKdV mapping. This we do by directly calculating the integrating factor that comes with each integral.

In terms of Θ's the integrals of the MKdV mapping are

$$I_q^p = \alpha_1 \left(v_0 v_p \Theta_{q-1,0}^{0,p-1} + \frac{1}{v_0 v_p} \Theta_{q-1,1}^{0,p-1} \right) + \alpha_2 \Theta_{q,1}^{0,p-1} + \alpha_3 \Theta_{q,0}^{0,p-1}.$$

The integrating factors are

$$\Lambda_q^p = \frac{1}{v_0 v_1 v_p v_{p+1}} \Theta_{q-1,1}^{1,p-1} - \Theta_{q-1,0}^{1,p-1}.$$

Theorem 1. *For any even positive q less than $p+1$ the expression I_q^p is a non-trivial integral of the MKdV mapping.*

Proof. We show that $S\left(I_q^p\right) - I_q^p = f_0 \Lambda_q^p$. The α_1-coefficient of I_q^p has the form $c + c^\tau$ with $c = v_0 v_p \Theta_{q-1,0}^{0,p-1}$. We have, using (3) and (4),

$$\begin{aligned} c &= v_0 v_p \Theta_{q-1,0}^{1,p-1} + \frac{v_p}{v_0} \Theta_{q-2,1}^{1,p-1} \\ &= v_0 v_p \Theta_{q-1,0}^{0,p-2} + \frac{v_0}{v_{p-1}} \Theta_{q-2,0}^{0,p-2}. \end{aligned}$$

Hence

$$S(c) = v_1 v_{p+1} \Theta^{1,p-1}_{q-1,0} + \frac{v_1}{v_p} \Theta^{1,p-1}_{q-2,0}$$

and we find that

$$\begin{aligned} S(c + c^\tau) - (c + c^\tau) &= (v_1 v_{p+1} - v_0 v_p) \Theta^{1,p-1}_{q-1,0} + \left(\frac{1}{v_1 v_{p+1}} - \frac{1}{v_0 v_p} \right) \Theta^{1,p-1}_{q-1,1} \\ &= (v_0 v_p - v_1 v_{p+1}) \Lambda^p_q. \end{aligned}$$

For the coefficient of α_2, $d = \Theta^{0,p-1}_{q,1}$, we have

$$\begin{aligned} d &= \Theta^{1,p-1}_{q,1} + v_0 v_1 \Theta^{1,p-1}_{q-1,0} \\ &= \Theta^{0,p-2}_{q,1} + \frac{1}{v_0 v_1} \Theta^{0,p-2}_{q-1,1}, \end{aligned}$$

and therefore

$$S(d) = \Theta^{1,p-1}_{q,1} + \frac{1}{v_0 v_1} \Theta^{1,p-1}_{q-1,1}.$$

We obtain

$$S(d) - d = v_0 v_1 \Lambda^p_q.$$

For the coefficient of α_3, d^τ, we obtain

$$\begin{aligned} S(d^\tau) - d^\tau &= \left(v_0 v_1 \Lambda^p_q \right)^\tau \\ &= v_p v_{p+1} \Theta^{1,p-1}_{q-1,0} - \frac{1}{v_0 v_1} \Theta^{1,p-1}_{q-1,1} \\ &= -v_p v_{p+1} \Lambda^p_q. \end{aligned}$$

Taking these all together we have shown that $S\left(I^p_q\right) - I^p_q = f_0 \Lambda^p_q$. Finally we note that $I^p_0 = \alpha_2 + \alpha_3$, $I^{2p-1}_{2p} = 2$ and $I^{p-1}_{q>p} = 0$ are trivial invariants. □

4. The staircase method

A PΔE on a two-dimensional lattice

$$f_{l,m} = f(v_{l,m}, v_{l+1,m}, v_{l,m+1}, v_{l+1,m+1}, \ldots) = 0, \tag{5}$$

with $l, m \in \mathbb{Z}$, has a Lax representation if there are matrices L, M, N depending on a spectral parameter k such that

$$L_{l,m} M^{-1}_{l,m} - M^{-1}_{l+1,m} L_{l,m+1} = f_{l,m} N_{l,m}, \tag{6}$$

in which $f_{l,m}$ does not depend on k, and $N_{l,m}$ is nonsingular on the equation [8]. Note that the right-hand side vanishes for solutions of the equation and is set to 0 by many authors. Similarly, an OΔE

$$f_n = f(v_n, v_{n+1}, v_{n+p}, v_{n+p+1}, \ldots) = 0, \tag{7}$$

with $n \in \mathbb{Z}$, has a Lax representation if there are matrices $\mathcal{L}, \mathcal{M}, \mathcal{N}$ depending on a spectral parameter k such that

$$\mathcal{M}_n \mathcal{L}_n - \mathcal{L}_{n+1} \mathcal{M}_n = f_n \mathcal{N}_n. \tag{8}$$

A PΔE can be reduced to an OΔE through a travelling wave reduction [7]. In the $(1, p)$ travelling wave reduction, which we consider in this paper, the equation and the Lax matrices depend on the independent variables l, m via the similarity variable $n = l + pm$. Thus we consider periodic solutions of the PΔE (5) satisfying $v_{l+p,m-1} = v_{l,m} = v_n$, which can be obtained from the OΔE (7).

The staircase method [6, 7] provides a way of generating invariants for OΔEs obtained in this way. The monodromy matrix $\mathcal{L}$ is defined to be the ordered product of Lax matrices along a standard staircase [7]. In the $(1, p)$ reduction the monodromy matrix is

$$\mathcal{L}_n = M_n^{-1} \overset{\curvearrowleft}{\prod_{j=n}^{n+p-1}} L_j, \qquad \text{where} \qquad \overset{\curvearrowleft}{\prod_{j=a}^{b}} L_j := L_b L_{b-1} \cdots L_a.$$

With $\mathcal{M}_n = L_n$ the Lax representation (6) for the PΔE (5) reduces to the Lax representation (8) for the OΔE (7), where

$$\mathcal{N}_n = N_n \overset{\curvearrowleft}{\prod_{j=n}^{n+p-1}} L_j.$$

Multiplying (8) by $-\mathcal{M}_n^{-1}$ and taking the trace we obtain that the trace of the monodromy matrix is invariant, that is

$$\operatorname{tr}(\mathcal{L}_{n+1}) - \operatorname{tr}(\mathcal{L}_n) = f_n \Lambda_n,$$

with the integrating factor

$$\Lambda_n = \operatorname{tr}\left(-N_n \overset{\curvearrowleft}{\prod_{j=n+1}^{n+p-1}} L_j \right). \tag{9}$$

One can expand the trace of the monodromy matrix in powers of the spectral parameter. Each coefficient will be an integral of the mapping. And the corresponding integrating factors will be the coefficients of the k-expansion of Λ_n.

Both the MKdV OΔE and the sine-Gordon OΔE introduced earlier are derived from integrable PΔEs by imposing the $(1, p)$ periodicity condition and they inherit a Lax representation. The sine-Gordon OΔE (2) is obtained from a generalization of the sine-Gordon PΔE [7, equation (2)]. The (generalized) reduced Lax matrices for the sine-Gordon equation are, cf [12],

$$\tilde{M}_n^{-1} = \begin{pmatrix} \beta_1 \frac{v_n}{v_{n+p}} & -\beta_3 k^{-2} \frac{1}{v_{n+p}} \\ -\beta_2 v_n & \beta_1 \end{pmatrix}, \qquad \tilde{L}_n = \begin{pmatrix} 1 & -v_{n+1} \\ -k^2 \frac{1}{v_n} & \frac{v_{n+1}}{v_n} \end{pmatrix}.$$

As one easily verifies the reduction of (6) is $\tilde{L}_n \tilde{M}_n^{-1} - \tilde{M}_{n+1}^{-1} \tilde{L}_{n+p} = \tilde{f}_n \tilde{N}_n$, with

$$\tilde{N}_n = \begin{pmatrix} \frac{1}{v_{n+p} v_{n+p+1}} & 0 \\ 0 & -\frac{1}{v_n v_{n+p}} \end{pmatrix}.$$

As was done for the sine-Gordon equation in [12] one can add parameters in the Lax matrices for the MKdV equation [7, equation A.1]. After an extra gauge transformation we obtain the following reduced Lax matrices,

$$M_n^{-1} = \begin{pmatrix} \alpha_3 \frac{v_{n+p}}{v_n} & k \\ k & \alpha_2 \frac{v_n}{v_{n+p}} \end{pmatrix}, \qquad L_n = \begin{pmatrix} \alpha_1 \frac{v_n}{v_{n+1}} & k \\ k & \alpha_1 \frac{v_{n+1}}{v_n} \end{pmatrix}.$$

We have $L_n M_n^{-1} - M_{n+1}^{-1} L_{n+p} = f_n N_n$, with

$$N_n = \begin{pmatrix} 0 & \frac{k}{v_{n+1} v_{n+p}} \\ -\frac{k}{v_n v_{n+p+1}} & 0 \end{pmatrix}.$$

Since we have expressed our integrals and integrating factors in terms of the variables $v_0, v_1, \ldots, v_p$, in the sequel we set $n = 0$, i.e. we consider $\mathrm{tr}(\mathcal{L}_0)$, and Λ_0. In subsequent sections we will show (theorem 3) that for the MKdV equation the trace of the monodromy matrix is

$$\mathrm{tr}(\mathcal{L}_0) = \sum_{i=0}^{\lceil p/2 \rceil} I_{2i}^p \alpha_1^{p-2i} k^{2i}.$$

We also show (theorem 4) that the integrating factor for the MKdV equation is expanded as

$$\Lambda_0 = \sum_{i=1}^{\lceil p/2 \rceil} \Lambda_{2i}^p \alpha_1^{p-2i} k^{2i}.$$

We obtained similar results for the sine-Gordon equation, the trace of its monodromy matrix $\tilde{\mathcal{L}}_0$ is

$$\mathrm{tr}(\tilde{\mathcal{L}}_0) = \sum_{i=0}^{\lfloor p/2 \rfloor} \tilde{I}_{2i}^p k^{2i},$$

and the integrating factor $\tilde{\Lambda}_0$ of the sine-Gordon equation is

$$\tilde{\Lambda}_0 = \sum_{i=0}^{\lfloor p/2 \rfloor} \tilde{\Lambda}_{2i}^p k^{2i}.$$

5. The monodromy matrix

In this section we show that the integrals I_q^p of the MKdV equation are the coefficients in the k-expansion of the trace of its monodromy matrix.

First we give the formula for the expansion of a product of L-matrices.

Lemma 2. *Let $p \geqslant a \in \mathbb{N}$. We have*

$$\prod_{j=a}^{\overset{\curvearrowleft}{p-1}} L_j = \sum_{i=0}^{p-a} Z_i^{a,p} \alpha_1^{p-a-i} k^i,$$

with

$$Z_i^{a,p} = \begin{pmatrix} \frac{v_a}{v_p} \Theta_{i,0}^{a,p-1} & 0 \\ 0 & \frac{v_p}{v_a} \Theta_{i,1}^{a,p-1} \end{pmatrix}$$

when i is even and

$$Z_i^{a,p} = \begin{pmatrix} 0 & \frac{1}{v_a v_p} \Theta_{i,1}^{a,p-1} \\ v_a v_p \Theta_{i,0}^{a,p-1} & 0 \end{pmatrix}$$

when i is odd.

Proof. We expand $L_n = \alpha_1 K_n + Jk$. It is plain that the k-degree of $L_{p-1} L_{p-2} \cdots L_a$ is $p - a$. Hence we have

$$\sum_{i=0}^{p-a} Z_i^{a,p} \alpha_1^{p-a-i} k^i = (\alpha_1 K_{p-1} + Jk) \sum_{i=0}^{p-a-1} Z_i^{a,p-1} \alpha_1^{p-a-i-1} k^i.$$

Equating coefficients we obtain the following recursive relations:

$$
\begin{aligned}
Z_0^{a,p} &= K_{p-1} Z_0^{a,p-1}\\
Z_i^{a,p} &= K_{p-1} Z_i^{a,p-1} + J Z_{i-1}^{a,p-1}, \qquad 0 < i < p-a\\
Z_{p-a}^{a,p} &= J Z_{p-a-1}^{a,p-1}.
\end{aligned}
$$

From the definition of $\Theta_{n,\epsilon}^{a,b}$ we have that $Z_0^{a,a}$ is the identity matrix. Thus, the lemma holds for $p = a, i = 0$. Assuming the result for $Z_0^{a,p-1}$ we can use the first recursive relation to show that

$$
\begin{aligned}
Z_0^{a,p} &= K_{p-1} Z_0^{a,p-1}\\
&= \begin{pmatrix} \frac{v_{p-1}}{v_p} & 0 \\ 0 & \frac{v_p}{v_{p-1}} \end{pmatrix} \begin{pmatrix} \frac{v_a}{v_{p-1}} & 0 \\ 0 & \frac{v_{p-1}}{v_a} \end{pmatrix}\\
&= \begin{pmatrix} \frac{v_a}{v_p} & 0 \\ 0 & \frac{v_p}{v_a} \end{pmatrix},
\end{aligned}
$$

which proves the lemma in the case $i = 0$. From the third recursive relation we obtain $Z_{p-a}^{a,p} = J^{p-a}$, in agreement with the lemma. Next, we proceed by induction on two variables, distinguishing the odd and even values of i. Assume that the lemma holds for $Z_i^{a,p-1}$ and $Z_{i-1}^{a,p-1}$. When i is odd we have

$$
\begin{aligned}
Z_i^{a,p} &= K_{p-1} Z_i^{a,p-1} + J Z_{i-1}^{a,p-1}\\
&= \begin{pmatrix} \frac{v_{p-1}}{v_p} & 0 \\ 0 & \frac{v_p}{v_{p-1}} \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{v_a v_{p-1}} \Theta_{i,1}^{a,p-2} \\ v_a v_{p-1} \Theta_{i,0}^{a,p-2} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{v_a}{v_{p-1}} \Theta_{i-1,0}^{a,p-2} & 0 \\ 0 & \frac{v_{p-1}}{v_a} \Theta_{i-1,1}^{a,p-2} \end{pmatrix}\\
&= \begin{pmatrix} 0 & \frac{1}{v_a v_p} \Theta_{i,1}^{a,p-2} + \frac{v_{p-1}}{v_a} \Theta_{i-1,1}^{a,p-2} \\ v_a v_p \Theta_{i,0}^{a,p-2} + \frac{v_a}{v_{p-1}} \Theta_{i-1,0}^{a,p-2} & 0 \end{pmatrix}\\
&= \begin{pmatrix} 0 & \frac{1}{v_a v_p} \Theta_{i,1}^{a,p-1} \\ v_a v_p \Theta_{i,0}^{a,p-1} & 0 \end{pmatrix},
\end{aligned}
$$

where the last equality holds after applying identity (4). When i is even we have

$$
\begin{aligned}
Z_i^{a,p} &= K_{p-1} Z_i^{a,p-1} + J Z_{i-1}^{a,p-1}\\
&= \begin{pmatrix} \frac{v_{p-1}}{v_p} & 0 \\ 0 & \frac{v_p}{v_{p-1}} \end{pmatrix} \begin{pmatrix} \frac{v_a}{v_{p-1}} \Theta_{i,0}^{a,p-2} & 0 \\ 0 & \frac{v_{p-1}}{v_a} \Theta_{i,1}^{a,p-2} \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{v_a v_{p-1}} \Theta_{i-1,1}^{a,p-2} \\ v_a v_{p-1} \Theta_{i-1,0}^{a,p-2} & 0 \end{pmatrix}\\
&= \begin{pmatrix} \frac{v_a}{v_p} \Theta_{i,0}^{a,p-2} + v_a v_{p-1} \Theta_{i-1,0}^{a,p-2} & 0 \\ 0 & \frac{v_p}{v_a} \Theta_{i,1}^{a,p-2} + \frac{1}{v_a v_{p-1}} \Theta_{i-1,1}^{a,p-2} \end{pmatrix}\\
&= \begin{pmatrix} \frac{v_a}{v_p} \Theta_{i,0}^{a,p-1} & 0 \\ 0 & \frac{v_p}{v_a} \Theta_{i,1}^{a,p-1} \end{pmatrix}.
\end{aligned}
$$

□

Now we can expand the trace of the monodomy matrix.

Theorem 3. *The trace of the monodromy matrix of the MKdV equation is*

$$\operatorname{tr}(\mathcal{L}_0) = \sum_{i=0}^{\lceil p/2\rceil} I_{2i}^p \alpha_1^{p-2i} k^{2i}.$$

Proof. We expand $M_0^{-1} = G_0\alpha_2 + H_0\alpha_3 + Jk$. Using lemma 2 we get

$$\begin{aligned}\operatorname{tr}(\mathcal{M}_0) &= \operatorname{tr}\sum_{i=0}^{p}(G_0\alpha_2 + H_0\alpha_3)Z_i^{0,p}\alpha_1^{p-i}k^i + \operatorname{tr}\sum_{i=1}^{p+1} JZ_{i-1}^{0,p}\alpha_1^{p-i+1}k^i \\ &= \sum_{i=1}^{p}\operatorname{tr}\left(G_0\alpha_2 Z_i^{0,p} + H_0\alpha_3 Z_i^{0,p} + JZ_{i-1}^{0,p}\alpha_1\right)\alpha_1^{p-i}k^i \\ &\quad + \operatorname{tr}\left(G_0\alpha_2 Z_0^{0,p} + H_0\alpha_3 Z_0^{0,p}\right)\alpha_1^p + \operatorname{tr}\left(JZ_p^{0,p}\right)k^{p+1}.\end{aligned} \tag{10}$$

We have $\operatorname{tr}\left(G_0\alpha_2 Z_0^{0,p} + H_0\alpha_3 Z_0^{0,p}\right) = \alpha_2 + \alpha_3 = I_0^p$ and

$$\operatorname{tr}(JZ_p^{0,p}) = \begin{cases} 0 & p \text{ is even} \\ 2 = I_{p+1}^p & p \text{ is odd.}\end{cases}$$

Hence, the k-degree of the trace of the monodromy matrix is $2\lceil p/2\rceil$. Also, the trace in the sum in (10) vanishes for odd values of i. When i is even we obtain

$$\begin{aligned}&\operatorname{tr}\left(G_0\alpha_2 Z_i^{0,p} + H_0\alpha_3 Z_i^{0,p} + JZ_{i-1}^{0,p}\alpha_1\right) \\ &\quad = \operatorname{tr}\left(\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & \frac{1}{v_0 v_p}\Theta_{i-1,1}^{0,p-1} \\ v_0 v_p \Theta_{i-1,0}^{0,p-1} & 0\end{pmatrix}\alpha_1\right. \\ &\qquad + \begin{pmatrix}0 & 0\\ 0 & \frac{v_0}{v_p}\end{pmatrix}\begin{pmatrix}\frac{v_0}{v_p}\Theta_{i,0}^{0,p-1} & 0 \\ 0 & \frac{v_p}{v_0}\Theta_{i,1}^{0,p-1}\end{pmatrix}\alpha_2 \\ &\qquad \left. + \begin{pmatrix}\frac{v_p}{v_0} & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\frac{v_0}{v_p}\Theta_{i,0}^{0,p-1} & 0 \\ 0 & \frac{v_p}{v_0}\Theta_{i,1}^{0,p-1}\end{pmatrix}\alpha_3\right) \\ &\quad = I_i^p.\end{aligned}$$

After a change of variable $i \mapsto 2i$ the result follows. □

6. The integrating factors revisited

In this section we show that the integrating factors for the MKdV equation arise as the coefficients in the k-expansion of Λ_0, as given by equation (9).

Theorem 4. *For the MKdV equation we have the expansion*

$$\Lambda_0 = \sum_{i=1}^{\lceil p/2\rceil} \Lambda_{2i}^p \alpha_1^{p-2i} k^{2i}.$$

Proof. Using lemma 2 with $a = 1$ we have

$$\begin{aligned}
\Lambda_0 &= \operatorname{tr}\left(-N_0 \overset{\curvearrowleft}{\prod_{j=1}^{p-1}} L_j\right) \\
&= -\operatorname{tr} \sum_{i=0}^{p-1} N_0 Z_i^{1,p} \alpha_1^{p-i-1} k^i, \\
&= -\operatorname{tr} \sum_{i=1}^{\lceil p/2 \rceil} \begin{pmatrix} 0 & \frac{1}{v_1 v_p} \\ -\frac{1}{v_0 v_{p+1}} & 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{v_1 v_p} \Theta_{i-1,1}^{1,p-1} \\ v_1 v_p \Theta_{i-1,0}^{1,p-1} & 0 \end{pmatrix} \alpha_1^{p-2i} k^{2i}, \\
&= \sum_{i=1}^{\lceil p/2 \rceil} \left(\frac{1}{v_0 v_1 v_p v_{p+1}} \Theta_{2i-1,1}^{1,p-1} - \Theta_{2i-1,0}^{1,p-1} \right) \alpha_1^{p-2i} k^{2i},
\end{aligned}$$

since the product $N_0 Z_i^{1,p}$ vanishes when i is even. □

7. Discussion

A major question is whether the maps studied in this paper are completely integrable in the Liouville-Arnold sense [2]. A discrete version of the Liouville-Arnold theorem, 5.1 in [11], tells us how many integrals of motion assure integrability of a given symplectic map, and describes the motion on the common level set of these integrals. For a $2N$-dimensional symplectic manifold one needs N functionally independent integrals that are in involution.

In this paper we have provided closed-form expressions for $\lfloor p/2 \rfloor$ integrals of the $(p + 1)$-dimensional MKdV mapping, and $\lceil p/2 \rceil$ integrals of the $(p + 1)$-dimensional sine-Gordon mapping. Fortunately, the dimension of the mappings can be reduced by certain transformations [7]. When p is even, the $(p + 1)$-dimensional MKdV map can be reduced by one dimension via the transformation $w_i = v_{i+1}/v_i$ leading to a p-dimensional map with $p/2$ explicit invariants. For p odd, the map can be reduced by two dimensions via the transformation $w_i = v_{i+2}/v_i$ leading to a $(p - 1)$-dimensional map with $(p - 1)/2$ explicit invariants. When p is odd, the $(p + 1)$-dimensional sine-Gordon map possesses $(p + 1)/2$ explicit invariants and, for p even, the mapping can be reduced by one dimension via the transformation $w_i = v_i v_{i+1}$ leading to a p-dimensional map with $p/2$ explicit invariants. In any case we have enough integrals. Explicit expressions for the symplectic structures of these reduced mappings were conjectured in [5]. Symplectic structures, and proofs, for these and for more general reductions can be found in [10].

We were able to verify the functional independence of all the invariants up to dimension 300. The involutivity with respect to the symplectic structures has been checked numerically up to dimension 20. With this evidence at hand it seems almost certain that the maps studied are integrable in the above sense. We have found nice generalizations of identities (3) and (4), which yield explicit formulae for the gradients of the integrals. They will play a role in the general proof, which we hope to publish elsewhere.

We have restricted our discussion to travelling-wave reductions of the lattice variables l, m via the similarity variable $n = l + pm$. It will be interesting to study more general reductions, considering $n = z_1 l + z_2 m$, $z_1, z_2 \in \mathbb{N}$. Another interesting path to pursue will be to consider ultra-discrete versions of the mappings considered.

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