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A novel *n*th order difference equation that may be integrable

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Abstract

We derive an *n*th order difference equation as a dual of a very simple periodic equation, and construct $\lfloor (n+1)/2 \rfloor$ explicit integrals and integrating factors of this equation in terms of multi-sums of products. We also present a generating function for the degrees of its iterates, exhibiting polynomial growth. In conclusion we demonstrate how the equation in question arises as a reduction of a system of lattice equations related to an integrable equation of Levi and Yamilov. These three facts combine to suggest the integrability of the *n*th order difference equation.

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1. Introduction

The notion of duality for difference equations was introduced in [1]. It was indicated there that integrable nonlinear equations often have integrable dual equations[4]. Sometimes dual equations have even more intricate structure than the primary equations. In this paper we explore this possibility starting with a somewhat trivial integrable equation: the *n*th order linear periodical equation: $u(l+n) = u(l)$, where l is the discrete variable and n is a fixed integer. For simplicity we re-write this equation employing subscripts:

$$u_{l+n} - u_l = 0. \tag{1}$$

Obviously, any function $I(u_l, \dots, u_{l+n-1})$ invariant under cyclic permutations is an integral of (1). In other words, for suitable functions I, the following relation is satisfied

$$\Delta I = (S-1)I = (u_{l+n} - u_l)\Lambda(u_l, \dots, u_{l+n}),$$

[4] Recently we discovered integrable maps whose duals do not seem to be integrable [12].

where S stands for the shift operator with respect to the independent variable: $S(a(l)) = a(l+1)$. The function Λ is called an integrating factor, and the equation $\Lambda = 0$ can be regarded as a dual to equation (1). Remarkably, as we will see below, a particular choice of integral leads to a seemingly integrable dual equation for all orders n. Another interesting aspect in this construction is the appearance of a combinatorial object known as a multi-sum of products. The latter seems to have first appeared in [2] as an ingredient in the construction of closed-form expressions for integrals of MKdV and sine-Gordon maps.

We take an integral of (1) in the form

$$I = \left(\prod_{q=0}^{n-1} u_{l+q}\right) + \zeta \left(\sum_{s=0}^{n-1} u_{l+s}\right)^{-1},$$

where ζ is an arbitrary constant. Let us find the dual equation for (1) corresponding to this integral. Differencing the above expression, we get

$$\Delta I = (u_{l+n} - u_l)\left(\left(\prod_{q=1}^{n-1} u_{l+q}\right) - \zeta\left(\sum_{s=1}^{n} u_{l+s}\sum_{p=0}^{n-1} u_{l+p}\right)^{-1}\right).$$

Therefore, the dual equation for (1) can be written as

$$\sum_{s=1}^{n} u_{l+s}\sum_{p=0}^{n-1} u_{l+p}\prod_{q=1}^{n-1} u_{l+q} = \zeta. \tag{2}$$

A more symmetric dual equation is found by interchanging integral I and constant ζ. We set $I = \alpha$, from which we obtain

$$\zeta = \sum_{s=0}^{n-1} u_{l+s}\left(\alpha - \prod_{q=0}^{n-1} u_{l+q}\right). \tag{3}$$

Note that the integrating factor corresponding to integral ζ is still $u_{l+n} - u_l$. Replacing ζ in (2) by the above expression we derive the equation

$$\sum_{s=0}^{n} u_{l+s}\prod_{q=1}^{n-1} u_{l+q} = \alpha. \tag{4}$$

The general construction of interchanging parameters and integrals introduced in [3] ensures that (3) is an integral of (4). The remainder of this paper will be focused on studying equation (4).

2. Integrals and multi-sums of products

Multi-sums of products were introduced in [2] in order to give closed-form expressions for integrals of mKdV and sine-Gordon maps. Further research revealed the ubiquity of this object [4, 5]. It was shown that integrals of travelling wave reductions of equations from the ABS list [6] can be expressed in terms of similar objects. In particular, the integrals of reductions of the lattice potential Korteweg–De Vries (pKdV) equation can be expressed in terms of certain functions Φ.

In this section, the multi-sums of products, Φ, are used to construct $\lfloor (n-1)/2 \rfloor$ additional integrals of (4). These functions are introduced in the following way. Suppose we have an ordered sequence of $N = b - a + 1$ variables

$$u_{l+a},\ u_{l+a+1},\ \ldots,\ u_{l+b-1},\ u_{l+b}, \tag{5}$$

where $a < b$ are some integers. Let us pick k different entries from the above sequence avoiding taking neighbouring ones, and form the products

$$u_{l+i_1}u_{l+i_2}\ldots u_{l+i_k}. \tag{6}$$

Thus, choosing $i_1 < i_2 < \ldots < i_k$, the integers i_j must satisfy the inequalities $i_{j+1} - i_j > 1$. Products (6) correspond to seating configurations with N seats and k persons who are not allowed to occupy neighbouring seats. The multi-sum of products $\Phi_k^{a,b}$ is defined as the sum of all such products. It follows directly from this definition that $\Phi_k^{a,b}$ is a homogeneous polynomial of order k, and that

$$\Phi_k^{a,b} = 0, \quad \text{for } k > (N+1)/2.$$

It is also worth noting that in the particular cases when $k = 0,\ k = 1$, we have

$$\Phi_0^{a,b} = 1, \quad \Phi_1^{a,b} = \sum_{i=a}^{b} u_{l+i}, \tag{7}$$

respectively. The following identities, involving multi-sums of products, Φ, will facilitate the construction of integrals of (4):

$$\Phi_k^{a,b} = \Phi_k^{a+1,b} + u_{l+a}\Phi_{k-1}^{a+2,b}, \qquad \Phi_k^{a,b} = \Phi_k^{a,b-1} + u_{l+b}\Phi_{k-1}^{a,b-2}. \tag{8}$$

Each of these identities is valid for arbitrary k. Indeed, if we collect the terms in $\Phi_k^{a,b}$ which do not contain u_{l+a}, we obtain $\Phi_k^{a+1,b}$. The sum of all terms that contain u_{l+a} can be written as $u_{l+a}F$, where F is a homogeneous polynomial of order $k-1$. This polynomial F cannot contain u_{l+1} as this would contradict the definition of Φ, therefore $F = \Phi_{k-1}^{a+2,b}$. Collecting the terms with and then without u_{l+b} in $\Phi_k^{a,b}$, we see that the second identity in (8) is also satisfied. We note that identities (8) are particular cases of more general recursive formulae

$$\begin{aligned}\Phi_k^{a,b} &= \sum_{i=0}^{k}\left(\Phi_{k-i}^{a,c-1}\Phi_i^{c+1,b} + \Phi_{k-i-1}^{a,c-2}\Phi_1^{c,c}\Phi_i^{c+2,b}\right)\\ &= \sum_{i=0}^{k}\left(\Phi_{k-i}^{a,c-1}\Phi_1^{c+1,c+1}\Phi_{i-1}^{c+3,b} + \Phi_{k-i}^{a,c}\Phi_i^{c+2,b}\right),\end{aligned}$$

where $a-1 \leqslant c \leqslant b+1$. For more detailed exposition on properties of multi-sum of products Φ, see [5].

Another identity, which will be used below, reads

$$\Phi_1^{0,n}\Delta\Phi_k^{1,n-2} = \Delta\left(\Phi_1^{0,n-1}\Phi_k^{1,n-2} - \Phi_{k+1}^{0,n-1}\right). \tag{9}$$

The proof is by direct calculation. Let us find the difference between left- and right-hand sides of (9):

$$\begin{aligned}\Phi_1^{0,n}\Delta\Phi_k^{1,n-2} - \Delta\left(\Phi_1^{0,n-1}\Phi_k^{1,n-2} - \Phi_{k+1}^{0,n-1}\right) &= \Phi_1^{0,n}\Delta\Phi_k^{1,n-2} - \Phi_1^{0,n-1}\Delta\Phi_k^{1,n-2}\\ &\quad - S(\Phi_k^{1,n-2})\Delta\Phi_1^{0,n-1} + \Delta\Phi_{k+1}^{0,n-1}\\ &= u_{l+n}\Delta\Phi_k^{1,n-2} - \Phi_k^{2,n-1}\Delta\Phi_1^{0,n-1} + \Delta\Phi_{k+1}^{0,n-1}.\end{aligned} \tag{10}$$

Here we used the formula $\Delta(ab) = \Delta(a)b + S(a)\Delta(b)$. Using the obvious identity

$$\Delta\Phi_k^{a,b} = \Phi_k^{a+1,b+1} - \Phi_k^{a,b},$$

we can rewrite (10) as

$$u_l\Phi_k^{2,n-1} - u_{l+n}\Phi_k^{1,n-2} + \Phi_{k+1}^{1,n} - \Phi_{k+1}^{0,n-1}. \tag{11}$$

Finally, due to formulae

$$\Phi_{k+1}^{0,n-1} = \Phi_{k+1}^{1,n-1} + u_l \Phi_k^{2,n-1}, \qquad \Phi_{k+1}^{1,n} = \Phi_{k+1}^{1,n-1} + u_{l+n} \Phi_k^{1,n-2},$$

which follow directly from (8), expression (11) is zero.

The last identity involving Φ to be mentioned here is

$$\frac{\Delta \Phi_k^{1,n-2}}{\prod_{s=1}^{n-1} u_{l+s}} = -\Delta\left(\frac{\Phi_{k-1}^{2,n-3}}{\prod_{s=1}^{n-2} u_{l+s}}\right). \tag{12}$$

We have

$$-\text{rhs} = \Delta\left(\frac{\Phi_{k-1}^{2,n-3}}{\prod_{s=1}^{n-2} u_{l+s}}\right) = \frac{\Phi_{k-1}^{3,n-2}}{\prod_{s=2}^{n-1} u_{l+s}} - \frac{\Phi_{k-1}^{2,n-3}}{\prod_{s=1}^{n-2} u_{l+s}} = \frac{u_{l+1}\Phi_{k-1}^{3,n-2} - u_{l+n-1}\Phi_{k-1}^{2,n-3}}{\prod_{s=1}^{n-1} u_{l+s}}.$$

Applying formulae (8) written in the reverse direction to the above expression, we obtain

$$-\text{rhs} = \frac{\Phi_k^{1,n-2} - \Phi_k^{2,n-2} - (\Phi_k^{2,n-1} - \Phi_k^{2,n-2})}{\prod_{s=1}^{n-1} u_{l+s}} = -\frac{\Delta \Phi_k^{1,n-2}}{\prod_{s=1}^{n-1} u_{l+s}} = -\text{lhs}.$$

Combining formulae (9) and (12), we obtain

$$\left(\Phi_1^{0,n} \prod_{s=1}^{n-1} u_{l+s} - \alpha\right) \frac{\Delta \Phi_k^{1,n-2}}{\prod_{s=1}^{n-1} u_{l+s}} = \Delta\left(\Phi_1^{0,n-1}\Phi_k^{1,n-2} - \Phi_{k+1}^{0,n-1} + \alpha \frac{\Phi_{k-1}^{2,n-3}}{\prod_{s=1}^{n-2} u_{l+s}}\right).$$

We have, therefore, proved the following:

Theorem. *In addition to integral (3), equation (4) admits the following integrals*

$$I_k = \Phi_1^{0,n-1}\Phi_k^{1,n-2} - \Phi_{k+1}^{0,n-1} + \alpha \frac{\Phi_{k-1}^{2,n-3}}{\prod_{s=1}^{n-2} u_{l+s}}, \qquad k = 1, \ldots, \lfloor (n-1)/2 \rfloor. \tag{13}$$

The corresponding integrating factors have the form

$$\Lambda_k = \frac{\Delta \Phi_k^{1,n-2}}{\prod_{s=1}^{n-1} u_{l+s}}.$$

Example. Let us write explicitly the integrals for the case $n = 3$. Equation (4) takes the form

$$u_{l+3} = -u_l - u_{l+1} - u_{l+2} + \frac{\alpha}{u_{l+1}u_{l+2}}.$$

For this equation we have the polynomial integral given by (3):

$$I_0 = \zeta = (u_l + u_{l+1} + u_{l+2})(\alpha - u_l u_{l+1} u_{l+2}),$$

and one rational integral given by (13):

$$I_1 = (u_l + u_{l+1} + u_{l+2})u_{l+1} - u_l u_{l+2} + \frac{\alpha}{u_{l+1}}.$$

Remark 1. We have proved that the integrals given by formulae (3) and (13) are functionally independent; please see the appendix for details.

Remark 2. When n is odd, integral (3) can be rewritten in terms of multi-sums of products, Φ, in the following way:

$$\zeta = GH,$$

where

$$G = \Phi_1^{0,n-1}\Phi_{(n-1)/2}^{1,n-2}, \qquad H = \frac{\alpha - \prod_{i=0}^{n-1} u_{l+i}}{\Phi_{(n-1)/2}^{1,n-2}}.$$

Both H and G are 2-integrals of equation (4), i.e. they satisfy $S^2(G) = G$ on solutions of (4). Moreover, they are related as $S(G) = H$, $S(H) = G$. This implies that the function $G + H$ is an integral and $G - H$ an anti-integral (i.e. $S(G - H) = H - G$) of (4). This, however, does not provide us with a new integral as $G + H = I_{(n-1)/2}$.

3. Algebraic entropy

Another characteristic property of integrable difference equations is that they seem to have vanishing algebraic entropy:

$$\lim_{k\to\infty} \frac{\ln d_k}{k} = 0,$$

where d_k is the sequence of degrees of iterates. The notion of algebraic entropy was introduced in [7, 8]; see also [9] and [10]. In fact, it has been observed that for integrable systems the degrees grow polynomially (of degree at most 2 or 3), implying zero entropy.

We have checked the behaviour of the sequences d_k for the maps of orders $n = 2, 3, 4, \ldots, 10$. In all these cases $d_k(n)$ grows as a second degree polynomial in k. It has been previously observed that generating functions for many discrete equations are rational functions with integer coefficients. For the maps we studied, we have found remarkably simple generating functions for the sequences, which seem to have the same form for arbitrary order n.

Taking initial conditions $u_l = t$ and u_{l+i} $(0 < i < n)$ randomly chosen integers, and $n \leqslant 10$, it is sufficient to perform 30 iterations or less to reveal a generating function. Note that the value of constant α can be set to 1 by rescaling u_l—this speeds up the calculation of d_k.

Sequences d_k corresponding to a few different n are given by

$$\begin{aligned}
&n = 2, \quad \{d_k\} = 1, 2, 3, 6, 9, 12, 17, 22, 27, 34, 41, \ldots \\
&n = 3, \quad \{d_k\} = 1, 2, 3, 5, 8, 11, 14, 18, 23, 28, 33, \ldots \\
&\ \ldots \\
&n = 10, \{d_k\} = 1, 2, 3, 5, 7, 9, 11, 13, 15, 17, 19, 22, 25, 28, 32, 36, 40, 44, 48, 52, 56, 60, \\
&\qquad\qquad\qquad 65, 70, 75, 81, 87, \ldots.
\end{aligned}$$

One can check that the generating function corresponding to these sequences is

$$f_n(x) = -\frac{x^3 + 1}{(x-1)^2(x^{n+1} - 1)},$$

that is, the Taylor series of f_n equals $f_n(x) = \sum_k d_k(n)x^k$. This function has a third order pole at $x = 1$ which confirms the quadratic growth of d_k [11]. Another choice of the initial conditions, for example $u_{l+1} = t$ and the other u_{l+i} randomly chosen, will result in a different generating function still having a third order pole at $x = 1$.

A second approach is to consider the second differences of the above sequences. The fact that the second differences are periodic shows the sequences have quadratic growth, and averaging yields the asymptotic behaviour

$$d_k(n) \sim \frac{1}{n+1} k^2.$$

4. Concluding remarks

We have presented an equation which is characterized by vanishing algebraic entropy and the presence of a normally sufficient number of integrals. However, the proof of integrability in the sense of Liouville–Arnold also requires the presence of a Poisson structure. Unfortunately, the general form of a Poisson structure remains unknown apart from a few lower-order cases ($n = 2, 3$). Other important properties of this equation include: the corresponding n-dimensional mapping is reversible and volume preserving but orientation-reversing in odd dimensions.

Many known integrable maps are obtained as reductions of partial difference equations like the ABS equations (see e.g. [4, 5]). Here we show how equation (4) is related to a reduction of a system of lattice equations, which in turn is related, by a Miura transformation, to equation (31) in [13]. However, it remains unknown how the Lax pair from Levi–Yamilov's paper can be employed to derive integrals of (4).

If we define $v_l = u_l + u_{l+1} + \cdots + u_{l+n}$ then

$$v_{l+1} - v_l = u_{l+n+1} - u_l. \tag{14}$$

Similarly, defining $w_l = u_{l+1}u_{l+2}\cdots u_{l+n-1}$ gives

$$\frac{w_{l+1}}{w_l} = \frac{u_{l+n}}{u_{l+1}}.$$

Using equation (4), which is $v_l w_l = \alpha$ to eliminate w we get

$$\frac{v_l}{v_{l+1}} = \frac{u_{l+n}}{u_{l+1}}. \tag{15}$$

The system of equations (14) and (15) is the $(n, -1)$-reduction [14] of the lattice system

$$v_{k+1,m} - v_{k,m} = u_{k+1,m+1} - u_{k,m}, \qquad \frac{v_{k,m}}{v_{k+1,m}} = \frac{u_{k,m+1}}{u_{k+1,m}}, \tag{16}$$

which acquires the form of a quotient-difference system; however, it is qualitatively different from *the* quotient-difference system [15]. Introducing a field w such that

$$v_{k,m} = \beta(m)w_{k-1,m}w_{k,m}, \qquad u_{k,m} = \beta(m)w_{k-1,m}w_{k,m} + w_{k,m}$$

the system reduces to a single equation

$$w_{k,m}(\beta(m)w_{k+1,m} + 1) = w_{k+1,m+1}(\beta(m+1)w_{k,m+1} + 1),$$

which is equation (31) in [13].

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Appendix. Functional independence of integrals

Since the set of integrals given by (3) and (13) consists of polynomial and rational functions, it is sufficient to prove that the corresponding Jacobian matrix has a full rank at the point $\mathbf{u} = (u_{l+i}) = (1, \ldots, 1) = \mathbf{1}$.

Lemma A1. *Let $a \leqslant b$ and $0 \leqslant 2r \leqslant b - a + 2$; then, we have*

$$\Phi_r^{a,b}\big|_{\mathbf{u}=\mathbf{1}} = \binom{b-a+2-r}{r}. \tag{A.1}$$

We first prove the case where $a = 1$ using induction with respect to b. It is easy to see that with $b = 1$, we have

$$\Phi_0^{1,1}\big|_{\mathbf{u}=\mathbf{1}} = 1 = \binom{1-1+2-0}{0}, \quad \text{and} \quad \Phi_1^{1,1}\big|_{\mathbf{u}=\mathbf{1}} = \binom{1-1+2-1}{1}.$$

With $b = 2$, we have

$$\Phi_0^{1,2}\big|_{\mathbf{u}=\mathbf{1}} = 1 = \binom{2-1+2-0}{0},$$

$$\Phi_1^{1,2}\big|_{\mathbf{u}=\mathbf{1}} = (u_{l+1} + u_{l+2})\big|_{\mathbf{u}=\mathbf{1}} = 2 = \binom{2-1+2-1}{1}.$$

Suppose the formula (A.1) holds for $b = n - 1$ and n ($n \geqslant 2$). We prove that it holds for $b = n + 1$. Using (2), we get

$$\Phi_r^{1,n+1} = \Phi_r^{1,n} + u_{l+n+2}\Phi_{r-1}^{1,n-1}.$$

Therefore, with $u_i = 1$ we have

$$\Phi_r^{1,n+1}\big|_{\mathbf{u}=\mathbf{1}} = \binom{n-(r-1)}{r-1} + \binom{n+1-r}{r} = \binom{n+2-r}{r} = \binom{n+1-1+2-r}{r}.$$

This means formula (A.1) holds for $b = n + 1$. Using the above result, we obtain

$$\Phi_r^{a,b}\big|_{\mathbf{u}=\mathbf{1}} = \Phi_r^{1,b-a+1}\big|_{\mathbf{u}=\mathbf{1}} = \binom{b-a+2-r}{r}.$$

Note that lemma 1 still holds if we define

$$\binom{n}{r} = \begin{cases} 0 & \text{if } r < 0 \text{ or } r > n, \\ 1 & \text{if } n \geqslant 0 \text{ and } r = 0, \\ \frac{n!}{r!(n-r)!} & \text{if } n \geqslant k \text{ and } r > 0 \end{cases} \tag{A.2}$$

for the cases where $2r < 0$ or $2r > b - a + 2$.

Now we can calculate the gradient of Φ using the recurrence

$$\Phi_k^{a,b} = \sum_{i=0}^{k-1} \Phi_{k-i-1}^{a,c-2}\Phi_i^{c+2,b}u_{l+c} + \sum_{i=0}^{k} \Phi_{k-i}^{a,c-1}\Phi_i^{c+1,b}, \tag{A.3}$$

where $a \leqslant c \leqslant b$ and $k \geqslant 0$, and formula (A.1). We have

$$\frac{\partial\Phi_k^{a,b}}{\partial u_{l+c}} = \sum_{i=0}^{k-1} \Phi_{k-i-1}^{a,c-2}\Phi_i^{c+2,b}. \tag{A.4}$$

The above expression when evaluated at $\mathbf{u} = \mathbf{1}$ is

$$\left.\frac{\partial\Phi_k^{a,b}}{\partial u_{l+c}}\right|_{\mathbf{u}=\mathbf{1}} = \sum_{i=0}^{k-1} \binom{c-a-k+\mathrm{i}+1}{k-\mathrm{i}-1}\binom{b-c-\mathrm{i}}{\mathrm{i}} \tag{A.5}$$

$$= \binom{b-a+2-k}{k} - \sum_{i=0}^{k} \binom{c-a-k+\mathrm{i}+1}{k-\mathrm{i}}\binom{b-c-\mathrm{i}+1}{\mathrm{i}}. \tag{A.6}$$

Therefore, we obtain the following.

Corollary. *Let* $0 < k < \lfloor \frac{n-1}{2} \rfloor = d$ *and* $2 \leqslant r \leqslant n - 2$. *We have*

$$\left.\frac{\partial I_0}{\partial u_{l+i}}\right|_{\mathbf{u}=\mathbf{1}} = \alpha - n - 1, \quad 0 \leqslant \mathrm{i} \leqslant n-1,$$

$$\left.\frac{\partial I_k}{\partial u_l}\right|_{\mathbf{u}=\mathbf{1}} = 0,$$

$$\left.\frac{\partial I_k}{\partial u_{l+1}}\right|_{\mathbf{u}=\mathbf{1}} = -(\alpha - n - 1)\binom{n-2-k}{k-1},$$

$$\left.\frac{\partial I_k}{\partial u_{l+r}}\right|_{\mathbf{u}=\mathbf{1}} = -(\alpha - n - 1)\sum_{i=0}^{k-1}\binom{r-k+\mathrm{i}}{k-i-1}\binom{n-2-r-\mathrm{i}}{\mathrm{i}}.$$

Denote the entries of the Jacobian matrix as

$$J_{ij} := \partial I_i/\partial u_{l+j}, \quad i, j = 0, \dots, d. \tag{A.7}$$

All expressions below are evaluated at the point $\mathbf{u} = 1$, so we omit the symbol $|_{\mathbf{u=1}}$.

Theorem A1. *Let $n > 4$, then the matrix J given by (A.7) can be factorized as*

$$J = -(\alpha - n - 1)\, LU,$$

where L and U are respectively lower and upper triangular matrices with the entries:

$$L_{ij} = \begin{cases} 0 & \text{if } i < j, \\ 0 & \text{if } j = 1, i > 0 \\ \binom{n-1-i-j}{i-j} & \text{otherwise,} \end{cases} \quad \text{and} \quad U_{ij} = \begin{cases} 0 & \text{if } i > j, \\ (-1)^i & \text{otherwise.} \end{cases}$$

In order to prove this theorem, we first prove

Lemma A2. *Let r and k satisfy the inequalities $2 \leqslant r \leqslant \lfloor (n+1)/2 \rfloor$ and $0 \leqslant k \leqslant n$, then the following formula is valid*

$$\frac{\partial \Phi_k^{1,n}}{\partial u_{l+r}} = \sum_{i=2}^{r+1} (-1)^i \binom{n+2-k-\mathrm{i}}{k+1-\mathrm{i}}. \tag{A.8}$$

The proof proceeds by induction on n where $n \geqslant 3$. We denote $T(n, k, r)$ and $S(n, k, r)$ the left- and right-hand sides of (A.8), respectively. One can check that the identity (A.8) holds for $n = 3$, $n = 4$ and $n = 5$. Suppose that this identity holds for $n - 1$ and n where $r \leqslant \lfloor n/2 \rfloor$, $0 \leqslant k \leqslant n-1$ and $2 \leqslant r \leqslant \lfloor (n+1)/2 \rfloor$, $0 \leqslant k \leqslant n-1$, respectively. We need to prove that it holds for $n+1$ with $2 \leqslant r \leqslant \lfloor (n+2)/2 \rfloor$ and $0 \leqslant k \leqslant n+1$.

Using the recurrence for Φ we obtain

$$\frac{\partial \Phi_k^{1,n+1}}{\partial u_{l+r}} = \frac{\partial \Phi_k^{1,n}}{\partial u_{l+r}} + u_{l+n+1} \frac{\partial \Phi_{k-1}^{1,n-1}}{\partial u_{l+r}}. \tag{A.9}$$

Therefore, we have

$$T(n+1, k, r) = T(n, k, r) + T(n-1, k-1, r). \tag{A.10}$$

It is also easy to see that the right-hand side of (A.8) satisfies the same recurrence, i.e.

$$S(n+1, k, r) = S(n, k, r) + S(n-1, k-1, r). \tag{A.11}$$

Therefore, using induction on n we obtain $T(n+1, k, r) = S(n+1, k, r)$ if $r \leqslant \lfloor n/2 \rfloor$ and $k \leqslant n$.

Now we have to prove that $T(n+1, k, r) = S(n+1, k, r)$ for $k = n+1$ or $r = \lfloor (n+2)/2 \rfloor$.

If $k = n+1$, we have $T(n+1, k, r) = 0$ as $\Phi_k^{1,n+1} = 0$ and also $S(n+1, k, r) = 0$ as

$$\binom{n+2-k-\mathrm{i}}{k+1-\mathrm{i}} = 0 \ \text{ for } \ i \geqslant 2.$$

Thus, we get $T(n+1, k, r) = S(n+1, k, r)$.

If $r = \lfloor (n+2)/2 \rfloor$, then one can see that $S(n+1, k, r) = 0$ and $T(n+1, k, r) = 0$ whenever $2k > n+2$. In the case where $2k \leqslant n+2$, i.e. $k \leqslant r$, we have

$$\Phi_k^{1,n+1} = \Phi_k^{2,n+1} + u_{l+1} \Phi_{k-1}^{3,n+1}.$$

Using the formula

$$\frac{\partial \Phi_{k-1}^{3,n+1}}{\partial u_{l+r}} = \frac{\partial \Phi_{k-1}^{1,n-1}}{\partial u_{l-2+r}} \text{ and } \frac{\partial \Phi_k^{2,n+1}}{\partial u_{l+r}} = \frac{\partial \Phi_k^{1,n}}{\partial u_{l-1+r}}$$

for $r > 3$ we obtain $T(n+1,k,r) = T(n,k,r-1) + T(n-1,k-1,r-2)$. In the case where $r = 3$, i.e. $n = 4$ or $n = 5$, one can verify that $S(n+1,k,r) = T(n+1,k,r)$.

Employing induction by n for $r > 3$, we get

$$\begin{aligned}T(n+1,k,r) &= \sum_{i=2}^{r}(-1)^i\binom{n+2-k-\mathrm{i}}{k+1-\mathrm{i}} + \sum_{i=2}^{r-1}(-1)^i\binom{n+2-k-\mathrm{i}}{k-\mathrm{i}}\\ &= \sum_{i=2}^{r}(-1)^i\binom{n+3-k-\mathrm{i}}{k+1-\mathrm{i}} - (-1)^r\binom{n+2-k-r}{k-r}.\end{aligned}$$

Hence, for $k < r$ we have

$$T(n+1,k,r) = \sum_{i=2}^{r}(-1)^i\binom{n+3-k-\mathrm{i}}{k+1-\mathrm{i}} = \sum_{i=2}^{r+1}(-1)^i\binom{n+3-k-\mathrm{i}}{k+1-\mathrm{i}} = S(n+1,k,r),$$

and in the case when $k = r$, we get

$$\begin{aligned}T(n+1,k,r) &= \sum_{i=2}^{r}(-1)^i\binom{n+3-k-\mathrm{i}}{k+1-\mathrm{i}} - (-1)^r = \sum_{i=2}^{r+1}(-1)^i\binom{n+3-k-\mathrm{i}}{k+1-\mathrm{i}}\\ &= S(n+1,k,r).\end{aligned}$$

This proves our statement

Using lemma 2, one can easily see that

$$\begin{aligned}\sum_{i=0}^{k-1}\binom{r-k+\mathrm{i}}{k-\mathrm{i}-1}\binom{n-r-\mathrm{i}}{\mathrm{i}} &= \sum_{i=2}^{r+1}(-1)^i\binom{n+2-k-\mathrm{i}}{k+1-\mathrm{i}}\\ &= \sum_{i=2}^{\min(r+1,k+1)}(-1)^i\binom{n+2-k-\mathrm{i}}{k+1-\mathrm{i}}.\end{aligned} \tag{A.12}$$

We are now ready to prove theorem 1. Denote $M := -(\alpha - n - 1)\, LU$. One can show that

$$M_{ij} = \begin{cases}\alpha - n - 1, & \text{if } 1 \leqslant j \leqslant d+1, i = 1,\\ 0, & \text{if } 1 < i \leqslant d+1,\ j = 1,\\ -(\alpha-n-1)\binom{n-i-3}{i-2}, & \text{if } 1 < i \leqslant d+1, j = 2,\\ -(\alpha-n-1)\left(\sum_{r=2}^{\min(i,j)}(-1)^r\binom{n-1-r-i}{i-r}\right), & \text{otherwise,}\end{cases}$$

which equals to $J_{ij} = \partial I_{i-1}/\partial u_{j-1}$. This proves our statement.

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