Integrable Boundary Conditions for Quad Equations, Open Boundary Reductions, and Integrable Mappings

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In the context of integrable difference equations on quad-graphs, we introduce the method of open boundary reductions, as an alternative to the well-known periodic reductions, for constructing discrete integrable mappings and their invariants. The mappings are obtained from well-posed initial value problems for quad and boundary equations restricted to strips on $\mathbb{Z}^2$-lattices. The invariants are constructed using Sklyanin's double-row monodromy matrix. To establish its properties, we use the discrete zero curvature condition and boundary zero curvature condition, showing how the latter derives from the boundary consistency condition. We focus on the Adler–Bobenko–Suris classification and associated integrable boundary equations. Examples are given for the H1 and Q1($\delta = 0$) equations, leading to novel maps of the plane.

1 Introduction

Some of the early motivations for integrable ordinary and partial difference equations dealt with the space and/or time discretisation of integrable ordinary/partial differential equations with the view of exact numerical schemes. This area is now a well-established field of its own with ramifications well beyond the original motivations,

see for instance the monographs [7, 14, 18]. The concept of commuting flows in the continuous world finds an analogue in the discrete world known as multidimensional consistency (or consistency around the cube) [6, 28, 29] for a certain class of partial difference equations formulated on quad-graphs. There is a deep connection between this notion and that of the set-theoretical Yang–Baxter equation [15]. Many other important features of integrability, such as Lax pairs and Bäcklund transformations, are shared between the continuous and discrete worlds.

More recently, the idea of a boundary consistency condition was introduced in [11] and emerged from the introduction of the set-theoretical reflection equation [10, 12]. The latter is a companion of the set-theoretical Yang–Baxter equation, and its solutions are called reflections maps, in analogy with Yang–Baxter maps [37]. Similarly, the boundary consistency condition is a companion to the multidimensional consistency condition for quad equations and is used to define associated discrete integrable boundary equations.

Integrable maps are the discrete analogues of integrable Hamiltonian flows and are thus of central importance; cf. [36]. They have been well studied and many examples have been constructed over the years, for example, the McMillan [25] map and the QRT maps [32, 33]. A systematic scheme to obtain higher-dimensional discrete maps with (potentially) enough invariants to ensure (discrete) Liouville integrability [36] is the periodic reduction method, also known as the staircase method [24, 30]. It relies on the possibility to define a well-posed initial value problem for partial difference equations on quad-graphs [3, 22] (typically in the form of a staircase for the $\mathbb{Z}^2$-lattice case [21]) and to impose periodicity on this initial value problem. The construction in [30] employs the Lax pair of the partial difference equation at hand. Invariant functions are obtained from the associated monodromy matrix, which is a product of Lax-matrices along the staircase. The involutivity of the invariants and complete integrability were proved later in [9].

In this paper, we present a new scheme to produce integrable maps as well as a generating function for their invariants. It is based on the notion of integrable boundary conditions for partial difference equations on quad-graphs with a boundary. The boundary on a quad-graph naturally consists of triangular faces on which one defines a boundary equation, as is shown in Figure 1. The boundary equations are said to be integrable if the associated boundary consistency condition is satisfied [11].

A key tool in our construction of invariants is the *double-row monodromy matrix*, originally introduced by Sklyanin [34, 35], whose properties depend on the rep-
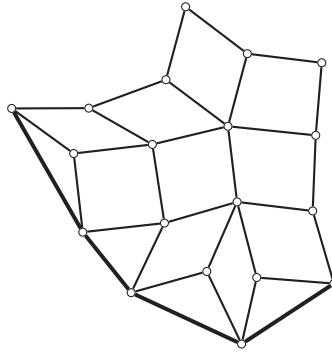
Fig. 1. A quad-graph with a boundary consisting of triangular faces.

resentation of integrable boundary equations via the *discrete boundary zero curvature equation* involving a *boundary matrix*.

We call the reductions obtained by our method *open boundary reductions*, in analogy, for instance, with the open boundary problems for the Toda lattice [27] and quantum spin chains [35]. They are constructed from a well-posed initial value problem on a quad-graph with boundary where integrable boundary equations are imposed. Note that there exists another type of reduction called singular boundary reduction [4]. In contrast to our construction, where boundary equations are formulated on triangles, the authors of [4] require the solutions to be singular on the edges of the boundary of a quad-graph.

The paper is organised as follows. In Section 2, we review and clarify the relevant ingredients involved in discrete integrable boundary conditions for equations on a quad-graph with a boundary. The key notion, introduced in [11], is the boundary consistency condition for the boundary equation. We show how this condition is related to a 2nd (dual) boundary consistency condition and subsequently to a zero curvature condition for the boundary equation. An important tool in doing so is the idea of folding explained in Section 2.3. Special emphasis is given to the ABS list [2] and its associated boundary equations. In Section 3, we construct nonautonomous and autonomous mappings as open boundary reductions of partial difference equations, from well-posed initial value problems on the $\mathbb{Z}^2$-lattice with two parallel boundaries. The boundary zero curvature representation of the integrable boundary conditions ensures (under certain conditions) the (anti-)isospectral property of the double-row monodromy matrix. This enables us to construct a generating function for (2-)integrals of the mappings. More generally, our construction provides us with k-integrals for

$(k+1)$-dimensional mappings; cf. [17]. In Section 4, we illustrate the results of Section 3 by explicitly computing some low-dimensional mappings and their invariants. The 3D nonautonomous open reduction of the Q1($\delta = 0$) equation gives rise to the following 2-parameter family of planar maps:

$$(x, y) \mapsto \frac{(x+y)(\alpha x + \beta(\alpha x + y + 1)y)^2}{\beta (x + (x\beta + y + 1)y)((\alpha^2 + \beta)xy^2 + \alpha(\beta x^2 + y^2)y + \alpha(x+y)^2)}$$
$$\left(\frac{x(\alpha x + (\alpha\beta x + \beta y + \alpha)y)}{\alpha (x + (\alpha x + y + 1)y)}, Y \right). \tag{1}$$

It leaves invariant the pencil of cubic curves (of genus 0):

$$y^2(1 + x + y) + \alpha x(\frac{x}{\beta} + xy + y^2) = Cxy. \tag{2}$$

Interestingly, the map can be understood geometrically as a composition of two Manin-involutions $\gamma = \iota_q \circ \iota_p$, where one involution point, for example, p is a non-singular base point of the pencil, but the other involution point, $q = q(C)$, depends on the particular curve in the pencil; cf. [23]. Section 5 contains concluding remarks and sketches examples of quad-graphs on a strip beyond the $\mathbb{Z}^2$-lattice case.

2   Boundary Equations, Folding, and Discrete Boundary Zero Curvature Conditions

In this section, we first recall the notions of 3D consistency and boundary consistency that are the integrability criteria for quad equations and their associated boundary equations respectively. Then, we introduce the notions of *dual boundary equation* and *dual boundary consistency condition*. We show that the dual boundary consistency condition gives rise to the *discrete boundary zero curvature condition*, which is the key structure in the open boundary reduction. This provides a systematic approach to deriving the *boundary matrices* appearing in the discrete boundary zero curvature condition. Explicit examples for the H1 and Q1($\delta = 0$) equations from the ABS list are provided.

2.1   3D-Consistent quad equations and discrete zero curvature conditions

Consider a quad equation, which is a partial difference equation defined on an elementary quadrilateral, as in Figure 2,

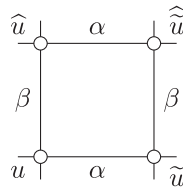$$Q(u, \widetilde{u}, \widehat{u}, \widehat{\widetilde{u}}; \alpha, \beta) = 0, \tag{3}$$

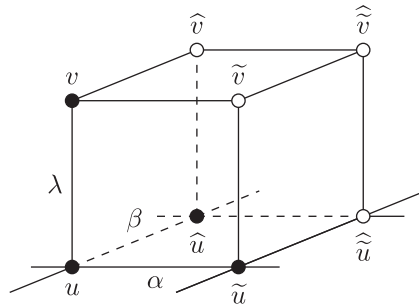**Fig. 2.** Elementary quadrilateral supporting the bulk equation.

Fig. 3. The 3D consistency: a quad equation can be imposed on the six faces of a cube. The black dots indicate the initial values.

where u is a discrete field defined on the underlying quad-graph. We employ the $\widetilde{}$, $\widehat{}$ notations to denote forward shifts of u along two independent directions and the lattice parameters α, β are associated with the $\widetilde{}$, $\widehat{}$ directions, respectively. For instance, if the underlying graph is the $\mathbb{Z}^2$-lattice,

$$u = u(n, m), \quad \widetilde{u} = u(n + 1, m), \quad \widehat{u} = u(n, m + 1), \quad \cdots. \tag{4}$$

The *bulk dynamics* on a quad-graph is determined by $Q = 0$ subject to well-posed initial data [3, 21]. A 3D consistency condition was introduced as a defining criterion for the integrability of quad equations [6, 28, 29]. Namely, a quad equation (3) is said to be integrable if it can be consistently defined on a cube; see Figure 3. Given the initial values $u, \widetilde{u}, \widehat{u}, v$ on a cube, the three possible ways to compute $\widehat{\widetilde{v}}$ must give the same result.

A classification of scalar 3D-consistent equations, known as the ABS list [2], was obtained under the following assumptions:

1. $\mathcal{D}_4$-symmetry, that is,

$$Q(u, \widetilde{u}, \widehat{u}, \widehat{\widetilde{u}}; \alpha, \beta) = \omega\, Q(\widetilde{u}, u, \widehat{\widetilde{u}}, \widehat{u}; \alpha, \beta) = \delta\, Q(u, \widehat{u}, \widetilde{u}, \widehat{\widetilde{u}}; \beta, \alpha), \quad \omega = \pm 1, \quad \delta = \pm 1. \tag{5}$$

The case where $\omega = -1$ is excluded; cf. [2].
2. Affine-linearity with respect to each of its fields.
3. The so-called tetrahedron property, that is, $\widehat{\widetilde{v}}$ does not depend on u but only on v, $\widetilde{u}$, and $\widehat{u}$ (and lattice parameters).

The 3D consistency, together with the above properties, allows one to derive discrete zero curvature conditions [6, 28]. It follows from $Q(u, \widetilde{u}, v, \widetilde{v}, \alpha, \lambda) = 0$ and the affine-linearity of Q that $\widetilde{v}$ can be expressed as a Möbius transformation acting on v:

$$\widetilde{v} = \frac{l_1\, v + l_2}{l_3\, v + l_4} = L\,[v], \quad L = L(\widetilde{u}, u; \alpha, \lambda) = \mu L_c, \tag{6}$$

where μ is a scalar function and the "core" of the matrix L is $L_c = \begin{pmatrix} l_1 & l_2 \\ l_3 & l_4 \end{pmatrix}$. Similarly, one has $\widehat{v} = M\,[v]$, $M = L(\widehat{u}, u; \beta, \lambda) = \nu M_c$. By composition of Möbius transformations, due to the 3D consistency, we get the discrete zero curvature condition

$$L(\widehat{\widetilde{u}}, \widetilde{u}; \beta, \lambda)\, L(\widetilde{u}, u; \alpha, \lambda)\,[v] = L(\widehat{\widetilde{u}}, \widehat{u}; \alpha, \lambda)\, L(\widehat{u}, u; \beta, \lambda)\,[v], \tag{7}$$

which holds when $Q(u, \widetilde{u}, \widehat{u}, \widehat{\widetilde{u}}; \alpha, \beta) = 0$. Note that since the lattice parameter α does not depend on the hat-shift and the lattice parameter β does not depend on the tilde-shift, (7) can be written as $\widetilde{M} L\,[v] = \widehat{L} M\,[v]$. Here, this zero curvature condition is formulated as a projective identity. The scalars μ, ν in the Lax matrices L, M are irrelevant in the action of the Möbius transformations. In order to obtain a true matrix equation

$$\widetilde{M} L = \widehat{L} M, \tag{8}$$

which holds when (3) is satisfied, one has to fix the scalars μ, ν appropriately. This is explained in detail in [8] for several classes of 3D-consistent equations. One option is to choose the normalisation so that the determinants of the Lax matrices equal 1. However, this may introduce unnecessary (square) roots to deal with. For this reason, we allow
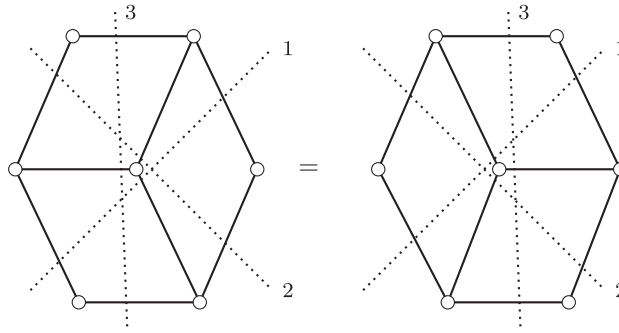
Fig. 4. Correspondence between the Yang–Baxter equation and the 3D consistency. The Yang–Baxter relation is represented by dotted lines, while the faces produce the face representation giving the two sides of the cube supporting the 3D consistency.

more freedom. For equations from the ABS list, the normalisation can be fixed such that

$$\det L(u, \widetilde{u}; \alpha, \lambda) = \ell(\alpha, \lambda) \quad L(u, \widetilde{u}; \alpha, \lambda)L(\widetilde{u}, u; \alpha, \lambda) = \pm\ell(\alpha, \lambda)\mathrm{id}, \tag{9}$$

where ℓ is a function depending only on the parameters. With appropriate normalisation, one can introduce an auxiliary vector Ψ on the elementary quadrilateral and interpret (8) as the compatibility condition, $\widehat{\widetilde{\Psi}} = \widetilde{\widehat{\Psi}}$, of the discrete linear Lax system

$$\widetilde{\Psi} = L\Psi, \qquad \widehat{\Psi} = M\Psi. \tag{10}$$

To prepare the ground for the formulation of the boundary consistency condition in the next subsection, it is convenient to recall the interplay between the graphical representations of the (quantum) Yang–Baxter equation and the consistency around the cube depicted in Figure 3. The face representation of the usual line representation of the Yang–Baxter equation produces the two sides of the relation as shown in Figure 4. For more details, see for instance [31].

2.2 Boundary consistency and integrable boundary equations

Inspired by [26], it was shown it [11] that by dualising a cellular decomposition of a surface with boundary, one naturally obtains a quad-graph where triangular faces represent the boundary. A *boundary equation* of the form

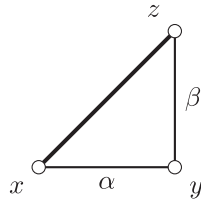$$q(x, y, z; \alpha, \beta) = 0, \tag{11}$$

Fig. 5. Elementary triangle supporting a boundary equation. The thick line represents the boundary connecting the boundary fields. This edge does not carry a parameter.

is required to hold on each elementary triangle, as in Figure 5, similarly to (3) holding on a quadrilateral.

By convention, the 1st and 3rd arguments in q, that is, x, z in (11), are the values of the field at the boundary vertices; the 2nd argument, that is, y in (11), is a bulk value of the field. The parameters α, β are the lattice parameters associated with edges connecting the bulk value y to the boundary values x and z, respectively.

It is natural to require boundary equations $q(x, y, z; \alpha, \beta) = 0$ to possess the following properties (i)–(iii).

(i) *Affine-linearity*: q is affine-linear with respect to the boundary fields:

$$q(x, y, z; \alpha, \beta) = m_1 xz + m_2 x + m_3 z + m_4 = q_1 z + q_2 = q_3 x + q_4, \qquad (12)$$

where $m_i = m_i(y; \alpha, \beta)$, $q_i = q_i(x, y; \alpha, \beta)$ when $i = 1, 2$ and $q_i = q_i(z, y; \alpha, \beta)$ when $i = 3, 4$.

(ii) *Nondegeneracy*: $q_{xz} q - q_x q_z \neq 0$. This amount to $m_1 m_4 - m_2 m_3 \neq 0$ and ensures that $q = 0$ can be solved to express x (resp. z) in terms of z (resp. x) and (possibly) y.

$$q(x, y, z; \alpha, \beta) = 0 \quad \Rightarrow \quad x = -\frac{q_4}{q_3} \text{ or } z = -\frac{q_2}{q_1}. \qquad (13)$$

(iii) $\mathbb{Z}_2$-*Symmetry*: there exists a function $h = h(\alpha, \beta)$ such that $q(x, y, z; \alpha, \beta) = h(\alpha, \beta) q(z, y, x; \beta, \alpha)$. For consistency, we have $h(\alpha, \beta) h(\beta, \alpha) = 1$.

In [11], a criterion, called *boundary consistency*, was introduced to select special boundary equations $q = 0$ yielding *integrable boundary conditions* for an integrable quad equation $Q = 0$. The criterion was not formalised into a definition in [11], but we do so now in this work.
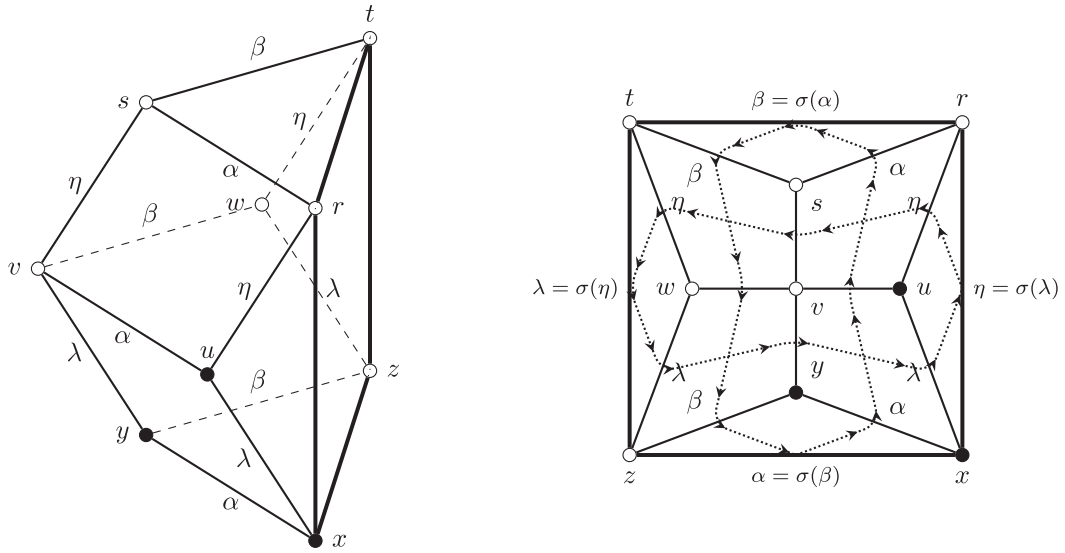
Fig. 6. Boundary consistency around half of a rhombic dodecahedron (left) and its planar projection (right), where $Q = 0$ is imposed on four quadrilaterals and $q = 0$ is imposed on four triangles. The (closed) characteristic lines (the dotted lines) are reflected at the boundary edges.

Definition 2.1. A boundary equation $q = 0$ is *boundary consistent* with an integrable quad equation $Q = 0$ if there is an involutive function σ between the parameters, $\beta = \sigma(\alpha)$ and $\eta = \sigma(\lambda)$, such that the initial value problem on the half-rhombic dodecahedron in Figure 6 is well-posed, that is, if the three ways of computing t from initial values x, y, u yield the same result. A boundary equation that is boundary consistent is called *integrable*.

Explicitly, the three ways of computing t alluded to in the definition mean the following (see right figure in Figure 6).

- Given the three initial values x, y, u, the equations

$$Q(x, y, u, v; \alpha, \lambda) = 0, \quad q(x, y, z; \alpha, \sigma(\alpha)) = 0, \quad q(x, u, r; \lambda, \sigma(\lambda)) = 0$$

  give the values of v, z, and r, respectively.
- Then, the equations

$$Q(u, v, r, s; \alpha, \sigma(\lambda)) = 0, \quad Q(y, z, v, w; \sigma(\alpha), \lambda) = 0$$

  give the values of s and w.

- Finally, the three equations

$$q(r, s, t; \alpha, \sigma(\alpha)) = 0, \quad q(z, w, t; \lambda, \sigma(\lambda)) = 0, \quad Q(v, w, s, t; \sigma(\alpha), \sigma(\lambda)) = 0$$

provide three ways of computing the value of t which we require to give the same answer.

A few comments are necessary for the reader unfamiliar with the historical development of the present ideas. First, one of the reasons behind calling a boundary equation, which is boundary consistent in the sense of Definition 2.1, integrable is that it ensures the possibility of constructing Bäcklund transformations as proved in [11, Proposition 1] based solely on the content of Definition 2.1. In the present work, what we show is that Definition 2.1 is the basis for the representation of integrable boundary equations in the form of a boundary zero curvature condition. The latter aspect is more important for our purpose of constructing invariants for mappings obtained by open boundary reductions; see Section 3.2.

Second, the origin of the half-rhombic dodecahedron as the natural structure to support the boundary consistency follows from the same reasoning that leads to 3D consistency from the (quantum) Yang–Baxter equation, which we briefly recalled at the end of the previous subsection. We take the *quantum reflection equation* [13, 35] as starting point; see Figure 7. It is established as the central object to characterise integrable boundaries for quantum integrable systems and as the companion to the quantum Yang–Baxter equation in that context. The main discovery of [12] is that it also appears naturally in the context of classical soliton theory and gives rise to set-theoretical solutions of the reflection equation, just like one can obtain set-theoretical solution of the Yang–Baxter equation from (vector) soliton collisions [1, 37]. From there, we consider the face representation associated with the line representation of the reflection equation (Figure 8) and obtain the half-rhombic dodecahedron as the resulting structure supporting the two sides; see [10] for more details.

Finally, the origin of the involution σ, which might seem a bit *ad hoc* here, actually follows the same historical development. Since the work of Cherednik and Sklyanin [13, 35], it has been known that the incorporation of integrable boundary conditions can be achieved by considering an involution acting on the spectral parameter (among other things). The precise map depends on the type of models (rational, trigonometric, etc.) but typical examples are $\lambda \mapsto -\lambda$ or $\lambda \mapsto 1/\lambda$. This is also what was obtained in [12] where the nonlinear mirror image method was implemented for vector solitons on the half-line, which helped axiomatise the results into the set-theoretical reflection equation.
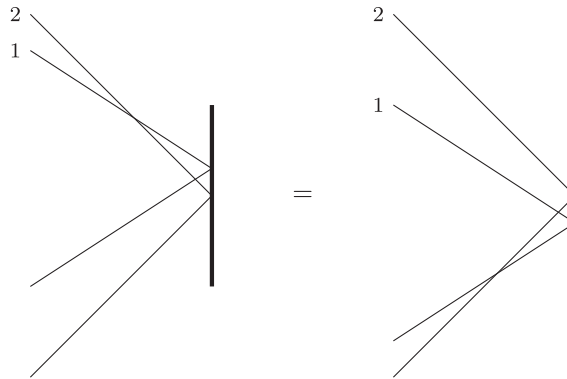
Fig. 7. Line representation of the reflection equation depicting the particle–particle interactions and particle–boundary interactions for two particles $1, 2$.
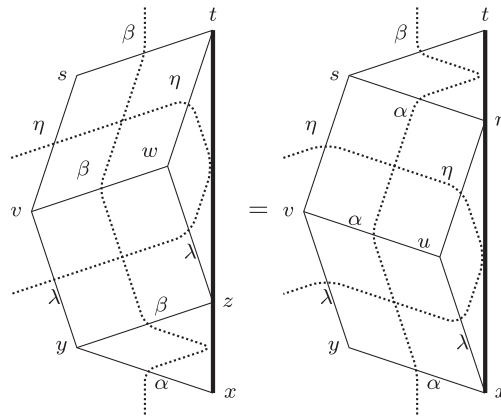

Fig. 8. Correspondence between the reflection equation and the boundary consistency. The reflection equation is represented by dotted characteristic lines, while the boundary consistency is represented by black faces. Combining the face representations of the reflection equation leads to a half-rhombic dodecahedron.

An involution acting on the spectral parameter naturally appeared in the construction. In the present context, σ plays this role and it acts naturally on the lattice parameters in Figure 6 according to the rule of Definition 2.1. A relation $\beta = \sigma(\alpha)$ necessarily satisfies $\sigma^2 = \mathrm{id}$ for characteristic lines that are closed due to the reflection at the boundary.

A list of integrable boundary equations for Q belonging to the ABS list was obtained in [11], using the idea of "folding the square into a triangle". In a nutshell, the idea is to look for q in the form of Q with one of the corner points eliminated in favour of two others via some unknown function. The ansatz is then inserted into the

equations for boundary consistency of Figure 6 in order to find the unknown function. More explicitly, one considers

$$Q(x, y, k(x, y, \alpha), z; \alpha, \sigma(\alpha)), \tag{14}$$

and looks for a function k (linear fractional in y) and an involution σ such that

(a)   the following factorisation holds:

$$Q(x, y, k(x, y, \alpha), z; \alpha, \sigma(\alpha)) = f(x, y, \alpha) q(x, y, z; \alpha), \tag{15}$$

with q having properties (i)–(iii) above;

(b)   the boundary consistency condition of Figure 6 hold.

The function k was a useful tool in [11] but its role was not fully exploited. In the next section, we elucidate the role of k as originating from a dual boundary equation associated with q. We also separate more clearly step (a), which contains no information about integrability but simply provides potential candidates for integrable boundary equations, and step (b), which deals with finding integrable boundary equations. The notion of dual boundary equation will enable us to formulate a novel (dual) boundary consistency condition.

2.3   Folding and dual boundary equations

We formalise the idea of folding as follows. Let an integrable quad equation $Q = 0$ be given. We will suppose a boundary equation $q = 0$, where q satisfies properties (i)–(iii), is given such that if we use $q(x, y, z; \alpha, \beta) = 0$ to express z as in (13) and eliminate z in $Q(x, y, c, z; \alpha, \beta)$, there exist a polynomial function $\chi(x, y; \alpha, \beta)$ and a polynomial $p(y, x, c; \alpha, \beta)$ *satisfying properties (i)–(iii)*, with the following relation holding:

$$q_1(x, y; \alpha, \beta) Q\left(x, y, c, -\frac{q_2(x, y; \alpha, \beta)}{q_1(x, y; \alpha, \beta)}; \alpha, \beta\right) = \chi(x, y; \alpha, \beta) \, p(y, x, c; \alpha, \beta). \tag{16}$$

The situation is illustrated in Figure 9.

We need to address to what extent the folding procedure provides a map from q to p. Let us note that if the factorisation (16) exists then p and χ are unique up to an overall function of the parameters α, β only. Therefore, strictly speaking to each q, we associate an equivalence class $[p]$ of boundary equations defined by the relation $p \sim p^*$ if and only $p(y, x, c; \alpha, \beta) = g(\alpha, \beta) p^*(y, x, c; \alpha, \beta)$ for some function $g(\alpha, \beta)$(note
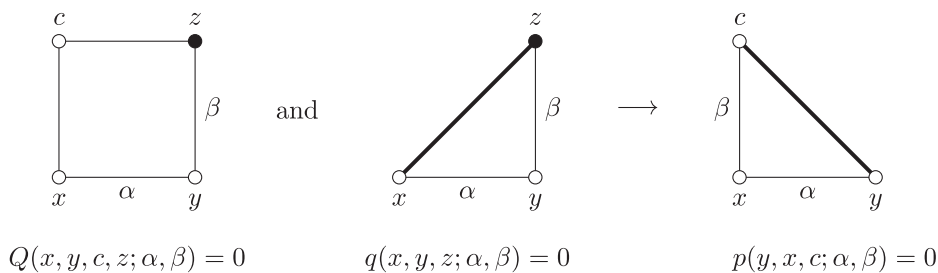
$$Q(x, y, c, z; \alpha, \beta) = 0 \qquad q(x, y, z; \alpha, \beta) = 0 \qquad p(y, x, c; \alpha, \beta) = 0$$

Fig. 9. Folding procedure: obtaining p from q and Q. We use $q = 0$ to eliminate z in Q (this elimination is shown by the black dots). β is independent of α at this stage, it will become $\sigma(\alpha)$ when considering integrability.
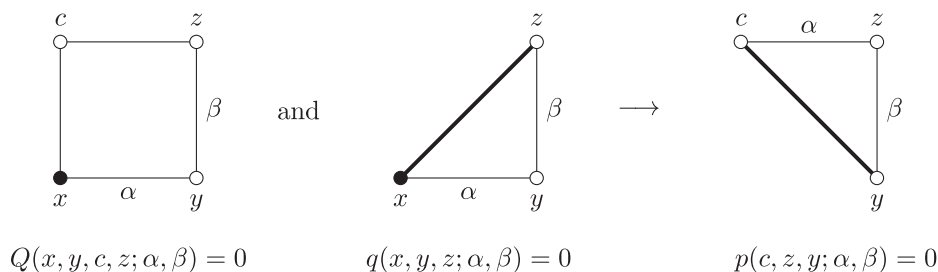

$$Q(x, y, c, z; \alpha, \beta) = 0 \qquad q(x, y, z; \alpha, \beta) = 0 \qquad p(c, z, y; \alpha, \beta) = 0$$

Fig. 10. Folding procedure: obtaining p from q and Q by eliminating x instead of z.

that the corresponding relation on χ, χ^* is $\chi^*(x, y; \alpha, \beta) = g(\alpha, \beta)\chi(x, y; \alpha, \beta))$. Of course, as far as the boundary equation $p = 0$ is concerned any representative p in $[p]$ yields the same relation on y, x, c. In particular, the same holds true for $q = 0$, so it is more appropriate to think of the folding as mapping an equivalence class $[q]$ to an equivalence class $[p]$. In practice, we can use any representative we like. In the rest of the paper, we will simply use the notation q and p as it should not lead to confusion whether we mean a representative or the class. Also, we will omit multipliers and dependence on variables/parameters when these are clear from the context, for example, equation (16) can be shortly written as $q_1 Q(x, y, c, -\frac{q_2}{q_1}; \alpha, \beta) \propto p(y, x, c; \alpha, \beta)$.

If the above folding occurs, then alternatively, we could decide to use $q(x, y, z; \alpha, \beta) = 0$ to eliminate x instead of z. This will give rise to *the same* boundary equation $p = 0$; see Figure 10.

Moreover, if one would use the newly obtained boundary equation $p = 0$ to eliminate either c or y, one will get back the original boundary equation $q = 0$.

Lemma 2.2 (Duality). Let $Q = Q(x, y, c, z; \alpha, \beta)$ be a multi-linear function with $\mathcal{D}_4$ symmetry, and let the boundary equation (12) possess properties (i)–(iii). Suppose that $p = p(y, x, c; \alpha, \beta)$ is a divisor of $q_1 Q(x, y, c, -\frac{q_2}{q_1}; \alpha, \beta)$ and that p also possesses properties (i)–(iii), so that p can be written variously as

$$p(y, x, c; \alpha, \beta) = p_1(y, x; \alpha, \beta)c + p_2(y, x; \alpha, \beta) = p_3(c, x; \alpha, \beta)y + p_4(c, x; \alpha, \beta). \tag{17}$$

Then, we have a) $q_3 Q(-\dfrac{q_4}{q_3}, y, c, z; \alpha, \beta) \propto p(c, z, y; \alpha, \beta),$

   b) $p_1 Q(x, y, -\dfrac{p_2}{p_1}, z; \alpha, \beta) \propto q(x, y, z; \alpha, \beta),$

   c) $p_3 Q(x, -\dfrac{p_4}{p_3}, c, z; \alpha, \beta) \propto q(z, c, x; \alpha, \beta).$

Proof.

   a) Because q has $\mathbb{Z}_2$ symmetry, apart from $x = -\dfrac{q_4(z, y; \alpha, \beta)}{q_3(z, y; \alpha, \beta)}$, we also have $x = -\dfrac{q_2(z, y; \beta, \alpha)}{q_1(z, y; \beta, \alpha)}$. Due to $\mathcal{D}_4$ symmetry, we have

$$q_3(z, y; \alpha, \beta)Q(-\frac{q_4(z, y; \alpha, \beta)}{q_3(z, y; \alpha, \beta)}, y, c, z; \alpha, \beta)$$

$$= h(\alpha, \beta)q_1(z, y; \beta, \alpha)\delta Q(z, y, c, -\frac{q_2(z, y; \beta, \alpha)}{q_1(z, y; \beta, \alpha)}; \beta, \alpha), \tag{18}$$

   which admits the divisor $p(y, z, c; \beta, \alpha) \propto p(c, z, y; \alpha, \beta)$.

   b) Since Q is a multivariate affine-linear polynomial, we write for convenience $Q(x, y, c, z; \alpha, \beta) = Q_1 cz + Q_2 c + Q_3 z + Q_4$, where $Q_j = Q_j(x, y; \alpha, \beta), j = 1, \ldots, 4$ are multivariate affine-linear polynomials in x, y. Substitution of $z = -\frac{q_2}{q_1}$ and multiplying by q_1 gives

$$-(Q_1 c + Q_3)q_2 + (Q_2 c + Q_4)q_1, \tag{19}$$

   which vanishes when $p(y, x, c; \alpha, \beta) = p_1 c + p_2 = 0$. Setting $p = 0$, expressing c in terms of x, y, and substituting in Q yields

$$p_1 Q(x, y, -\frac{p_2}{p_1}, z, \alpha, \beta) = -(Q_1 z + Q_2)p_2 + (Q_3 z + Q_4)p_1, \tag{20}$$

$q(z, c, x; \alpha, \beta) = 0$

$p(c, z, y; \alpha, \beta) = 0$

$$\Longleftrightarrow$$

$$Q(x, y, c, z; \alpha, \beta) = 0$$

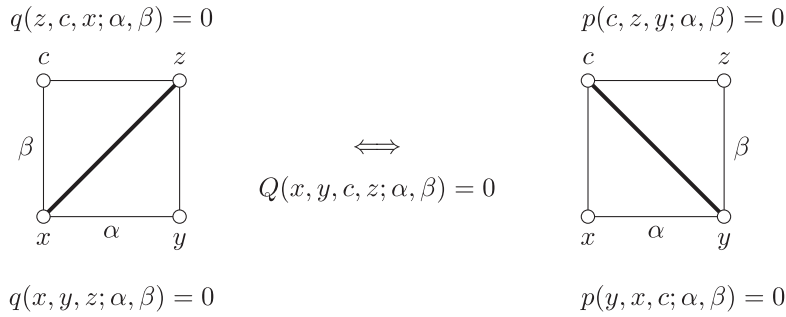$q(x, y, z; \alpha, \beta) = 0$

$p(y, x, c; \alpha, \beta) = 0$

Fig. 11. Folding of Q along the two diagonals: $p = 0$ is the dual of $q = 0$ and vice versa.

which is a multivariate polynomial in x, y, z and is linear in z. This polynomial vanishes for $z = -\frac{q_2}{q_1}$ in view of (19), so it must be proportional to $q_1 z + q_2 = q(x, y, z; \alpha, \beta)$.

c)   This follows from (b) using the $\mathcal{D}_4$ symmetry of Q and the $\mathbb{Z}_2$ symmetry of q, p.   ∎

Lemma 2.2 leads us to the following definition.

Definition 2.3.   If Q, q, p satisfy the conditions in Lemma 2.2, then we say that $p = 0$ is the *dual boundary equation* of $q = 0$ and vice versa. The quad equation $Q = 0$ is said to have a pair of dual boundary equations $p = 0$, $q = 0$.

The duality property is illustrated in Figure 11: if Q is folded by either $p = 0$ or $q = 0$, then it is folded into two copies of p through $q = 0$ and vice versa.

It is important to note that the folding procedure explained here does not contain information about integrability of the boundary equation $q = 0$. It is a 1st step towards selecting candidates for integrable qs. Going back to the original construction of [11] involving the function k as in (14), we see now that the latter is nothing but $-\frac{p_2}{p_1}$ obtained when eliminating c using $p = 0$. This function plays a crucial role in defining the boundary matrix K appearing in the boundary zero curvature equation (Section 2.4). The notion of a pair of dual boundary equations $q = 0$, $p = 0$ puts on firm ground the idea of folding of Q that was introduced in [11]. It is a valuable notion for (at least) two reasons: (1) it provides the precise link between the boundary consistency condition and the boundary zero curvature equation (see Proposition 2.5 below); (2) dual boundary equations provide good candidates for integrable boundary equations. All the integrable
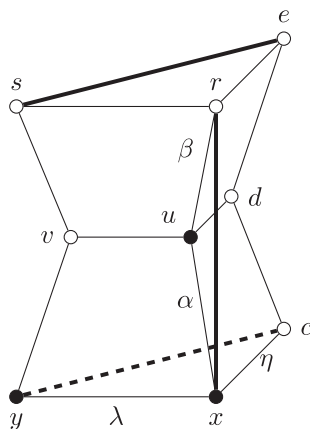
Fig. 12. Dual boundary consistency: the equation $q = 0$ is imposed on the vertical triangle and its dual $p = 0$ is imposed on the top and bottom triangles. The two ways of computing e from x, y, u lead to the same value.

boundary equations found in [11] fall into this category. In fact, many more integrable boundary equations for the ABS list can be obtained this way. This is left for future work.

2.4   Dual boundary consistency and the boundary zero curvature condition

A discrete boundary zero curvature condition, connecting a 3D-consistent equation and its integrable boundary equations, was formulated in [11]. In this section, we explain the connection between boundary consistency and the discrete boundary zero curvature condition. Equipped with the notion of dual boundary equation, we now proceed to formulate a different boundary consistency condition in terms of a dual pair (q, p).

Definition 2.4.   Let $Q = 0$ be an integrable quad equation, which admits a pair of dual boundary equations $q = 0$, $p = 0$. The ordered pair (q, p) is said to be *dual boundary consistent with $Q = 0$* if there is an involutive function σ between the parameters, $\beta = \sigma(\alpha)$ and $\eta = \sigma(\lambda)$, such that the initial value problem on the 3D-stencil in Figure 12, where $p = 0$ is imposed on the top and bottom triangles and $q = 0$ is imposed on the vertical triangle, is well-posed, that is, the two ways of computing e from initial values x, y, u yield the same value.

For boundary equations that admit a dual boundary equation, the above consistency condition turns out to be equivalent to the original consistency condition.

Proposition 2.5.   A boundary equation $q = 0$, with dual $p = 0$, is boundary consistent with $Q = 0$ according to Definition 2.1, if and only if the ordered pair (q, p) is dual boundary consistent with $Q = 0$ according to Definition 2.4.

Note that the proposition does not imply that $p = 0$ is boundary consistent if $q = 0$ is.

Proof.   For convenience, when $Q = 0$ is used to express one variable in terms of the other three (and the parameters), we will write for short $(x, y, u) \xrightarrow{Q} v$. Similarly, with $q = 0$, we write for instance $(x, u) \xrightarrow{q} r$. Recall that $\beta = \sigma(\alpha)$ and $\eta = \sigma(\lambda)$.

$\Rightarrow$ Consider the figure on the left in Figure 13, which represents the boundary consistency, where $q = 0$ is imposed on the four boundary triangles and $Q = 0$ is imposed on the four quadrilaterals. The values of the vertices z, v, w, r, s, t are consistently defined. We embed it into the middle figure by adding the vertices c, d, e, which are defined as follows: $(x, y, z) \xrightarrow{Q} c$, $(u, v, w) \xrightarrow{Q} d$, and $(r, s, t) \xrightarrow{Q} e$. The equation $Q = 0$ is imposed on the added quadrilaterals $(xcdu)$, $(czwd)$, $(uder)$, and $(dwte)$. The 3D consistency of $Q = 0$ ensures that d and e are defined uniquely and consistently. Finally, we move to the figure on the right by noting that on the bottom quadrilateral, we now have $q(x, y, z; \lambda, \eta) = 0$ and $Q(x, y, c, z; \lambda, \eta) = 0$, so that by the duality property, we have $p(y, x, c; \lambda, \eta) = 0$. This is indicated by the change of the dashed diagonal line from (xz) to (yc). Similarly, on the top quadrilateral, we have $p(s, r, e; \lambda, \eta) = 0$. From the point of view of the initial value problem, we can determine e consistently from y, x, u as follows:

$$(x, y, u) \xrightarrow{Q} v, \quad (x, u) \xrightarrow{q} r, \quad (u, v, r) \xrightarrow{Q} s, \quad (s, r) \xrightarrow{p} e$$

or

$$(x, y) \xrightarrow{p} c, \quad (c, x, u) \xrightarrow{Q} d, \quad (x, u) \xrightarrow{q} r, \quad (d, u, r) \xrightarrow{Q} e.$$

It remains to delete the vertices z, w, t to obtain precisely the dual boundary consistency of (q, p) with Q illustrated in Figure 12.

$\Leftarrow$ Starting from the dual boundary consistency diagram 12 and adding the vertices z, w, and t by using Q as before, we obtain the figure on the right hand in Figure 13 where all the vertices are consistently defined from x, y, u.
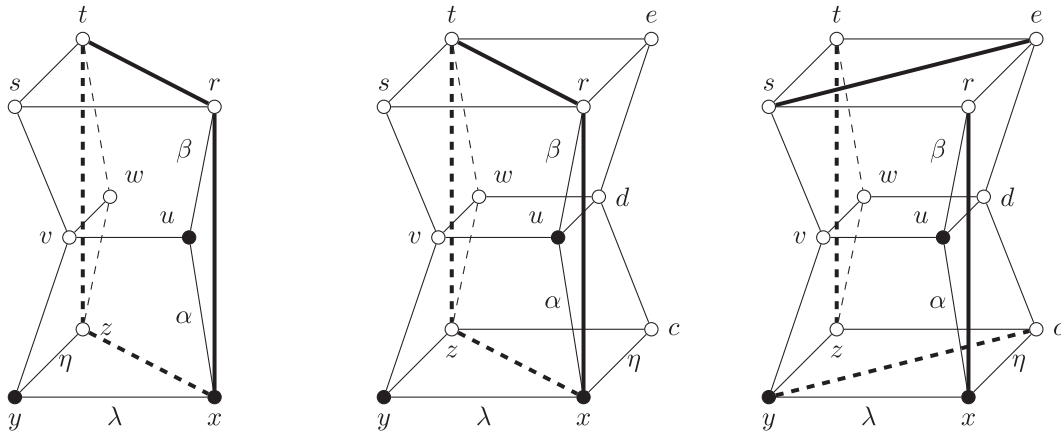
Fig. 13. Equivalence of boundary consistency and dual boundary consistency through the duality property.

The factorisation of Q implies we have $p(y, z, c; \eta, \lambda) = p(s, t, e; \eta, \lambda) = 0$ and the dual boundary consistency of (q, p) with $Q = 0$ implies that $q(z, w, t; \alpha, \beta) = 0$. We can switch the diagonals on the top and bottom faces due to the duality property between p and q. Lastly, we delete the vertices c, d, e to obtain the boundary consistency condition between q and Q, that is, the figure on the left-hand side, as desired. ∎

The significance of Proposition 2.5 is that, through the dual object p, we can relate the boundary consistency for q and Q with a discrete boundary zero curvature condition involving a boundary matrix K, in the same way as one can relate the 3D consistency of Q with the zero curvature condition (7).

Since integrable boundary equations only depend on one parameter, as the 2nd parameter $\eta = \sigma(\lambda)$ is related to the 1st by the involution σ, in the sequel, we write $q = q(x, y, z; \lambda)$, and similarly for p. As explained in Section 2.1, we can associate a Lax matrix L with Q. Similarly, we can associate a boundary matrix K to $p(y, x, c; \lambda) = 0$, by expressing c as a Möbius transformation acting on y:

$$c = \frac{k_1(x; \lambda)y + k_2(x; \lambda)}{k_3(x; \lambda)y + k_4(x; \lambda)} = K[y], \quad K = K(x; \lambda) = \kappa K_c, \tag{21}$$

where κ is a scalar function and $K_c = \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix}$ is the "core" of the *boundary matrix* K. The $\mathbb{Z}_2$ symmetry of p implies that $K_c(x; \lambda) K_c(x; \sigma(\lambda))$ is proportional to the identity matrix.

Using Figure 12, by composition of Möbius transformations, we get the following *projective* discrete boundary zero curvature condition showing the two ways of expressing e from y

$$L_c(r, u, \sigma(\alpha), \sigma(\lambda))\, L_c(u, x, \alpha, \sigma(\lambda))\, K_c(x, \lambda)\, [y] = K_c(r; \lambda)\, L_c(r, u; \sigma(\alpha), \lambda)\, L_c(u, x; \alpha, \lambda)\, [y], \tag{22}$$

which holds when $q(x, u, r; \alpha) = 0$, with $\beta = \sigma(\alpha)$; cf. property (iv) for q. As was the case for the bulk equation $Q = 0$, we want to write the boundary zero curvature condition as a matrix equation that should hold on $q = 0$. Letting $u = \widetilde{x}$ and $r = \widetilde{\widetilde{x}}$ and

$$L = L(\widetilde{x}, x; \alpha, \lambda), \quad M = L(\widehat{x}, x; \sigma(\alpha), \lambda), \tag{23}$$

then equation (22) can be written as $\widetilde{\widehat{M}}\, \overline{L}\, K\, [y] = \widehat{\widehat{K}}\, \widetilde{M}\, L\, [y]$, whose matrix version reads

$$\widetilde{\widehat{M}}\, \overline{L}\, K = \widehat{\widehat{K}}\, \widetilde{M}\, L. \tag{24}$$

Taking determinants, using normalisation (9) and setting for convenience $\rho = \sqrt{\det K} = \kappa \sqrt{\det K_c}$, we obtain the following condition relating the normalisation of L and that of K,

$$\left(\frac{\widehat{\widehat{\rho}}}{\rho} \right)^2 = \frac{\ell(\sigma(\alpha), \sigma(\lambda))\, \ell(\alpha, \sigma(\lambda))}{\ell(\sigma(\alpha), \lambda)\, \ell(\alpha, \lambda)}. \tag{25}$$

Note that the left-hand side is independent of α so can be evaluated at $\alpha = \alpha_0$ where α_0 is a fixed point of σ. This implies that

$$\frac{\widehat{\widehat{\rho}}}{\rho} = \epsilon \frac{\ell(\alpha_0, \sigma(\lambda))}{\ell(\alpha_0, \lambda)}, \quad \epsilon^2 = 1. \tag{26}$$

We can now re-scale L by a function of λ as

$$L(u, x; a, \lambda) \rightarrow \frac{L(u, x; a, \lambda)}{\sqrt{\ell(\alpha_0, \lambda)}}, \quad \ell(\alpha, \lambda) \rightarrow \frac{\ell(\alpha, \lambda)}{\ell(\alpha_0, \lambda)}, \tag{27}$$

to obtain

$$\widehat{\widehat{\rho}} = \epsilon\rho, \qquad \epsilon^2 = 1. \tag{28}$$

This suggests that K should be normalised such that ρ does not depend on the field x as otherwise (28) could lead to a relation between x and r, which is incompatible with $q(x, u, r; \alpha) = 0$. In practice, we normalise the matrix K such that

$$K(x; \lambda)\, K(x; \sigma(\lambda)) = \mathrm{id}. \tag{29}$$

The reason for not scaling L to have determinant 1 is to avoid unnecessary square roots that are hard to deal with in a computer algebra environment. Allowing ϵ to be ±1 is necessary for (24) to be equivalent to the boundary equation in general. Indeed, in some examples (see below), it can happen that $\epsilon = -1$ is required. With this in mind, and summarising our account, we will take

$$L(r, u; \sigma(\alpha), \sigma(\lambda))\, L(u, x; \alpha, \sigma(\lambda))\, K(x; \lambda) = \epsilon\, K(r; \lambda)\, L(r, u; \sigma(\alpha), \lambda)\, L(u, x; \alpha, \lambda), \quad \epsilon = \pm1, \tag{30}$$

as the boundary zero curvature representation of the integrable boundary equation $q(x, u, r; \alpha) = 0$ in the rest of this paper, with the understanding that L is normalised by applying the rescaling (27) to (9) and K is normalised as in (29), with a determinant independent of the field.

In general, both L and K may involve square roots of the parameters and/or fields in their expressions, which makes it difficult to extract invariants. However, in certain cases, the freedom we have exploited here makes it possible to deal with this. This will be made clear in the examples below.

2.5 Examples of integrable boundary equations and boundary matrices

We provide integrable boundary equations and their boundary matrices for the H1 and Q1($\delta = 0$) equations from the ABS list. These examples will be used in Section 4 in the construction of integrable mappings by the open reduction method. Note that for all integrable boundary equations obtained in [11], the involution σ is either in an "additive" form:

$$\sigma(\alpha) = -\alpha + 2\mu, \tag{31}$$

Table 1   Integrable boundary equations for H1

Boundary equation	Dual boundary equation		Boundary matrix
$q(x, y, z; \alpha) = 0$	$p(y, x, c; \alpha) = 0$	ϵ	$K(x; \lambda)$
$y(z - x) + \alpha - \mu = 0$	$y + c = 0$	1	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
$x + z = 0$	$x(y - c) + \mu - \alpha = 0$	-1	$\begin{pmatrix} 1 & \frac{\mu - \lambda}{x} \\ 0 & 1 \end{pmatrix}$

or in a "multiplicative" form:

$$\sigma(\alpha) = \frac{\mu^2}{\alpha}, \tag{32}$$

where μ is a free parameter in both cases ($\mu \neq 0$ in the multiplicative case). In each case, $\alpha_0 = \mu$ is a fixed point of σ which we use to implement (27).

In order to illustrate the procedure of normalisation for the boundary zero curvature equation, for each example, we provide the Lax matrix L corresponding to $Q = 0$, with the normalisation obtained in [8] and the corresponding function $\ell(\alpha, \lambda)$. Also, the determinants of the boundary matrices K are constant, and it is understood that, given σ as in (31) or (32), one should perform the rescaling (27).

2.5.1   *H1, case of additive σ*

The equation reads

$$(u - \widehat{\widetilde{u}})(\widetilde{u} - \widehat{u}) + \beta - \alpha = 0 \tag{33}$$

and has Lax matrix

$$L(\widetilde{u}, u; \alpha, \lambda) = \begin{pmatrix} u & \alpha - \lambda - u\widetilde{u} \\ 1 & -\widetilde{u} \end{pmatrix}, \quad \ell(\alpha, \lambda) = \lambda - \alpha, \tag{34}$$

Table 1 gives the required elements to obtain the boundary zero curvature representation of each equation $q = 0$. Here, $\sigma(\alpha) = -\alpha + 2\mu$, with μ being a free parameter appearing also in one of the integrable boundary equations.

Table 2   Examples of integrable boundary equations for Q1($\delta = 0$) under additive σ

Boundary equation	Dual boundary equation		Boundary matrix
$q(x, y, z; \alpha) = 0$	$p(y, x, c; \alpha) = 0$	ϵ	$K(x; \lambda)$
$\alpha(xz - y^2) - \mu(x - y)(y + z) = 0$	$y + c = 0$	1	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
$x + z = 0$	$\alpha(x^2 - cy) + \mu(c + x)(y - x) = 0$	-1	$\frac{1}{\lambda(2\mu - \lambda)} \begin{pmatrix} \mu & (\lambda - \mu)x \\ \frac{(\lambda - \mu)}{x} & \mu \end{pmatrix}$

2.5.2   *Q1($\delta = 0$), case of additive σ*

The equation reads

$$\alpha(u - \widehat{u})(\widetilde{u} - \widehat{\widetilde{u}}) - \beta(u - \widetilde{u})(\widehat{u} - \widehat{\widetilde{u}}) = 0, \tag{35}$$

with Lax matrix

$$L(\widetilde{u}, u; \alpha, \lambda) = \frac{1}{\widetilde{u} - u} \begin{pmatrix} \lambda(\widetilde{u} - u) - \alpha\widetilde{u} & \alpha\widetilde{u}u \\ -\alpha & \lambda(\widetilde{u} - u) + \alpha u \end{pmatrix}, \quad \ell(\alpha, \lambda) = \lambda(\lambda - \alpha). \tag{36}$$

Table 2 gives the required elements to obtain the boundary zero curvature representation of $q = 0$ for each example. Here, $\sigma(\alpha) = -\alpha + 2\mu$.

2.5.3   *Q1($\delta = 0$), case of multiplicative σ*

For convenience, let us consider another form of Q1($\delta = 0$)

$$\frac{1}{\alpha^2}(u - \widehat{u})(\widetilde{u} - \widehat{\widetilde{u}}) - \frac{1}{\beta^2}(u - \widetilde{u})(\widehat{u} - \widehat{\widetilde{u}}) = 0, \tag{37}$$

which has Lax matrix

$$L(\widetilde{u}, u; \alpha, \lambda) = \frac{1}{\alpha^2(\widetilde{u} - u)} \begin{pmatrix} \alpha^2(\widetilde{u} - u) - \lambda^2\widetilde{u} & \lambda^2\widetilde{u}u \\ -\lambda^2 & \alpha^2(\widetilde{u} - u) + \lambda^2 u \end{pmatrix}, \quad \ell(\alpha, \lambda) = 1 - \frac{\lambda^2}{\alpha^2}. \tag{38}$$

Here, $\sigma(\alpha) = \mu^2/\alpha$.

Table 3 Examples of integrable boundary equations for Q1($\delta = 0$) as in (37) with a multiplicative σ

Boundary equation	Dual boundary equation		Boundary matrix
$q(x, y, z; \alpha) = 0$	$p(y, x, c; \alpha) = 0$	ϵ	$K(x; \lambda)$
$\alpha^2(x - y) + \mu^2(y - z) = 0$	$\alpha^2(y - x) - \mu^2(x - c) = 0$	1	$\begin{pmatrix} -\frac{\lambda}{\mu} & \frac{(\lambda^2+\mu^2)x}{\lambda\mu} \\ 0 & \frac{\mu}{\lambda} \end{pmatrix}$
$\alpha^2(x - y) - \mu^2(y - z) = 0$	$\alpha^2(y - x) + \mu^2(x - c) = 0$	-1	$\begin{pmatrix} \frac{\lambda}{\mu} & \frac{(\mu^2-\lambda^2)x}{\lambda\mu} \\ 0 & \frac{\mu}{\lambda} \end{pmatrix}$
$\alpha^2(x-y)z + \mu^2(y-z)x = 0$	$\alpha^2(y-x)c - \mu^2(x-c)y = 0$	1	$\begin{pmatrix} -\frac{\mu}{\lambda} & 0 \\ -\frac{\lambda^2+\mu^2}{\lambda\mu x} & \frac{\lambda}{\mu} \end{pmatrix}$
$\alpha^2(x-y)z - \mu^2(y-z)x = 0$	$\alpha^2(y-x)c + \mu^2(x-c)y = 0$	-1	$\begin{pmatrix} \frac{\mu}{\lambda} & 0 \\ \frac{\mu^2-\lambda^2}{\lambda\mu x} & \frac{\lambda}{\mu} \end{pmatrix}$

3   Integrable Mappings from Open Boundary Reductions

We introduce the idea of open boundary reductions of quad equations on quad-graphs with two parallel boundaries as a new means to construct integrable mappings. The key ingredients are the well-posedness of the initial data as well as the boundary zero curvature conditions. A generating function for the invariants of these mappings will be obtained.

3.1   Open boundary reductions on the $\mathbb{Z}^2$-lattice

We consider quad-graphs with two parallel boundaries as depicted in Figure 14. On the "left" boundary (fields with index 1), we impose boundary conditions associated with $q_- = 0$, while on the "right" boundary (fields with index n), we impose boundary conditions associated with $q_+ = 0$. Given a 3D consistent quad equation $Q = 0$ imposed on the bulk (composed of quadrilaterals), $q_- = 0$ and $q_+ = 0$ can be different solutions to the boundary consistency condition but under the same σ. In order to describe the map, we denote the solution of $Q(u, \tilde{u}, \hat{u}, \hat{\tilde{u}}, \alpha, \beta) = 0$ with respect to $\hat{\tilde{u}}$ by $\hat{\tilde{u}} = F(u, \tilde{u}, \hat{u}, \alpha, \beta)$, and the solution of $q_\pm(x, y, z, \alpha) = 0$ with respect to z by $z = f_\pm(x, y, \alpha)$. In the simplest
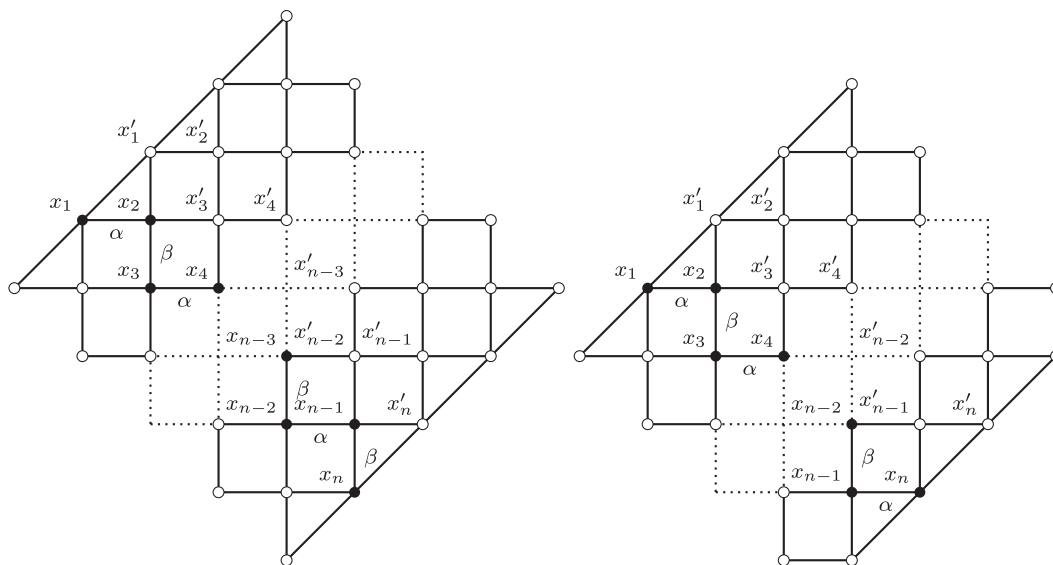
Fig. 14. A well-posed initial boundary-value problem on $\mathbb{Z}^2$-lattice on a strip: here, $\beta = \sigma(\alpha)$, and $x_1, \ldots, x_n$ (black dots) are the initial boundary data.

initial value problem we consider here, the initial data $x_1, \ldots, x_n$ and $\alpha_1, \ldots, \alpha_{n-1}$ will evolve to $x'_1, \ldots, x'_n$ and $\alpha'_1, \ldots, \alpha'_{n-1}$ by one-step discrete "time" and eventually propagate to infinity. This corresponds to a collective move of the fields and parameters from lattice site (n, m) to $(n+1, m+1)$ within the strip, following the notations of (4).

First, we take the lattice parameters to be α on horizontal edges and $\sigma(\alpha)$ on vertical edges. The so-constructed maps are *autonomous*, as the parameters remain unchanged after one-step of evolution.

In the graph on the left, we have an odd number, $n = 2k + 1$, of variables. The upward evolution (north-east direction in our figures) is given by

$$
\begin{cases}
x'_1 = f_-(x_1, x_2, \alpha), \\
x'_{2i+1} = F(x_{2i+1}, x_{2i+2}, x_{2i}, \alpha, \sigma(\alpha)), & 1 \le i < k, \\
x'_n = f_+(x_n, x_{n-1}, \sigma(\alpha)), \\
x'_{2i} = F(x_{2i}, x'_{2i+1}, x'_{2i-1}, \alpha, \sigma(\alpha)), & 1 \le i \le k.
\end{cases}
\tag{39}
$$

In the graph on the right, we have an even number, $n = 2k+2$, of variables. The evolution upwards is given by

$$\begin{cases} x_1' = f_-(x_1, x_2, \alpha), \\ x_{2i+1}' = F(x_{2i+1}, x_{2i+2}, x_{2i}, \alpha, \sigma(\alpha)), & 1 \le i \le k, \\ x_{2i}' = F(x_{2i}, x_{2i+1}', x_{2i-1}', \alpha, \sigma(\alpha)), & 1 \le i \le k, \\ x_n' = f_+(x_n, x_{n-1}', \sigma(\alpha)). \end{cases} \tag{40}$$

The inverses of these maps, that is, the downward evolution (south-west direction) can be written down in a similar fashion .

We can consider similar initial boundary-value problems, but with general lattice parameters along the staircase. Here, the maps need to be accompanied by an action on the parameters and hence become *nonautonomous*. As this action is cyclic, one can consider the $(n-1)$-th power of this map, which is again autonomous.

With general lattice parameters (Figure 15), we have the following maps. The 2D map is the same as (40). The 3D map is

$$\begin{cases} x_1' = f_-(x_1, x_2, \alpha_1), \\ x_3' = f_+(x_3, x_2, \alpha_2), \\ x_2' = F(x_2, x_3', x_1', \sigma(\alpha_2), \sigma(\alpha_1)), \end{cases} \tag{41}$$

with

$$(\alpha_1, \alpha_2) \to (\sigma(\alpha_2), \sigma(\alpha_1)). \tag{42}$$

The odd-dimensional map with $n = 2k + 1$, $k > 1$ is

$$\begin{cases} x_1' = f_-(x_1, x_2, \alpha_1), \\ x_{2i+1}' = F(x_{2i+1}, x_{2i+2}, x_{2i}, \alpha_{2i+1}, \alpha_{2i}), & 1 \le i < k-1, \\ x_n' = f_+(x_n, x_{n-1}, \alpha_{2k}), \\ x_2' = F(x_2, x_3', x_1', \alpha_3, \sigma(\alpha_1)), \\ x_{2i}' = F(x_{2i}, x_{2i+1}', x_{2i-1}', \alpha_{2i+1}, \alpha_{2i-2}), & 1 < i < k, \\ x_{n-1}' = F(x_{n-1}, x_n', x_{n-2}', \sigma(\alpha_{2k}), \alpha_{2k-2}), \end{cases} \tag{43}$$
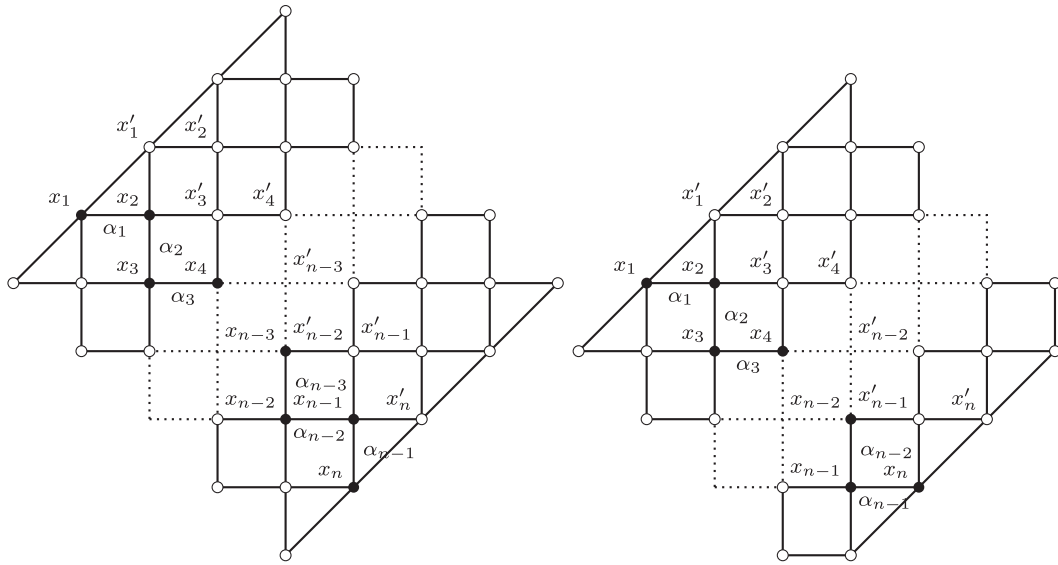
Fig. 15. More general case with $n - 1$ lattice parameters.

with

$$
\begin{cases}
\alpha_2 \to \alpha'_2 = \sigma(\alpha_1), \\
\alpha_{2i-1} \to \alpha'_{2i-1} = \alpha_{2i+1}, & 1 \le i < k, \\
\alpha_{2i+2} \to \alpha'_{2i+2} = \alpha_{2i}, & 1 \le i < k, \\
\alpha_{2k-1} \to \alpha'_{2k-1} = \sigma(\alpha_{2k}).
\end{cases}
\tag{44}
$$

The even-dimensional map with $n = 2k + 2$, $k > 0$ is

$$
\begin{cases}
x'_1 = f_-(x_1, x_2, \alpha_1), \\
x'_{2i+1} = F(x_{2i+1}, x_{2i+2}, x_{2i}, \alpha_{2i+1}, \alpha_{2i}), & 1 \le i \le k, \\
x'_2 = F(x_2, x_3, x_1, \alpha_3, \sigma(\alpha_1)), \\
x'_{2i} = F(x_{2i}, x'_{2i+1}, x'_{2i-1}, \alpha_{2i+1}, \alpha_{2i-2}), & 1 < i \le k, \\
x'_n = f_+(x_n, x'_{n-1}, \alpha_{2k})),
\end{cases}
\tag{45}
$$

with

$$
\begin{cases}
\alpha_2 \to \alpha_2' = \sigma(\alpha_1), \\
\alpha_{2i-1} \to \alpha_{2i-1}' = \alpha_{2i+1}, & 1 \le i < k, \\
\alpha_{2i+2} \to \alpha_{2i+2}' = \alpha_{2i}, & 1 \le i < k, \\
\alpha_{2k+1} \to \alpha_{2k+1}' = \sigma(\alpha_{2k}).
\end{cases}
\tag{46}
$$

Of course, the case with general lattice parameters contains the situation of Figure 14 as a particular case where $\alpha_{2i+1} = \alpha$ and $\alpha_{2i} = \sigma(\alpha)$.

3.2 Generating function for the invariants

Consider the maps defined above with general lattice parameters (Figure 15). Let us first recall the bulk monodromy matrix $T(\lambda)$ from x_1 to x_n as the following ordered product of Lax matrices L associated with the bulk equation $Q = 0$

$$
T(\lambda) = \prod_{j=1,\cdots,n-1}^{\frown} L(x_{j+1}, x_j; \alpha_j, \lambda) = L_{n,n-1}(\lambda) \dots L_{2,1}(\lambda).
\tag{47}
$$

Here, the notation $L_{j+1,j}(\lambda) = L(x_{j+1}, x_j; \alpha_j, \lambda)$ is understood. Note that $T(\lambda)$ depends on all the x_js and α_js, but we do not show this dependence explicitly for conciseness. The updated values of $T(\lambda)$, that is, its value at x_j' and α_j', will be simply denoted by $T'(\lambda)$. Similarly, let $\check{T}$ denote the reverse-ordered monodromy matrix from x_n to x_1

$$
\check{T}(\lambda) = \prod_{j=1,\cdots,n-1}^{\frown} L(x_j, x_{j+1}; \alpha_j, \lambda) = L_{1,2}(\lambda) \dots L_{n-1,n}(\lambda).
\tag{48}
$$

Now, let $n = 2k + 2$ for even n, and $n = 2k + 1$ for odd n. Inspired by Sklyanin's construction [34, 35], we now define the so-called double-row monodromy matrix $\mathcal{T}(\lambda)$ in the form

$$
\mathcal{T}(\lambda) = K_-(x_1; \sigma(\lambda)) \, \check{T}(\sigma(\lambda)) \, K_+(x_n; \lambda) \, T(\lambda),
\tag{49}
$$

where K_- satisfies

$$
L_{1',2}(\lambda) L_{2,1}(\lambda) K_-(x_1, \sigma(\lambda)) = \epsilon_- K_-(x_1', \sigma(\lambda)) L_{1',2}(\sigma(\lambda)) L_{2,1}(\sigma(\lambda)), \quad \epsilon_- = \pm 1,
\tag{50}
$$

corresponding to $q_-(x_1, x_2, x'_1; \alpha_1) = 0$ and K_+ satisfies

$$L_{n',n-1}(\sigma(\lambda))L_{n-1,n}(\sigma(\lambda))K_+(x_n, \lambda) = \epsilon_+ K_+(x'_n, \lambda)L_{n',n-1}(\lambda)L_{n-1,n}(\lambda), \quad \epsilon_+ = \pm 1, \tag{51}$$

corresponding to $q_+(x_n, x_{n-1}, x'_n; \alpha_{2k}) = 0$ if n is odd, that is, $n = 2k + 1$ or

$$L_{n',n-1'}(\sigma(\lambda))L_{n-1',n}(\sigma(\lambda))K_+(x_n, \lambda) = \epsilon_+ K_+(x'_n, \lambda)L_{n',n-1'}(\lambda)L_{n-1',n}(\lambda), \quad \epsilon_+ = \pm 1, \tag{52}$$

corresponding to $q_+(x_n, x'_{n-1}, x'_n; \alpha_{2k}) = 0$ if n is even, that is, $n = 2k + 2$. We have the following result.

Theorem 3.1.   Let L be the Lax matrix associated with $Q = 0$ from the ABS list, and K_- and K_+ be the boundary matrices associated with the integrable boundary equation $q_- = 0$ and $q_+ = 0$, normalised as explained in Section 2.4, that is, such that the boundary zero curvature conditions (50) and (51) (for odd $n = 2k + 1$) or (52) (for even $n = 2k + 2$) hold. Then, the monodromy matrix $\mathcal{T}(\lambda)$ defined in (49) and its updated version $\mathcal{T}'(\lambda)$ are related by(we stress that here $\ell(\alpha, \lambda)$ is the normalisation of L taking (27) into account)

$$\mathcal{T}'(\lambda)\mathcal{E} = \epsilon_- \epsilon_+ \frac{\ell(\sigma(\alpha_1), \lambda)\ell(\sigma(\alpha_{2k}), \sigma(\lambda))}{\ell(\alpha_1, \sigma(\lambda))\ell(\alpha_{2k}, \lambda)} \mathcal{E}\,\mathcal{T}(\lambda), \tag{53}$$

where

$$\mathcal{E} = L(x'_1, x_2; \sigma(\alpha_1), \lambda)L(x_2, x_1; \alpha_1, \lambda). \tag{54}$$

Proof.   Let us consider the odd case $n = 2k + 1$. The even case $n = 2k + 2$ is completely analogous. For simplicity, we use the notations $L_{j+1,j}(\lambda) = L(x_{j+1}, x_j; \alpha_j, \lambda)$, $L_{j+1,j'}(\lambda) = L(x_{j+1}, x'_j; \alpha_j, \lambda)$, etc., by dropping the dependence of the lattice parameter as it is always associated with the edge connecting the two adjacent vertices. It follows from (9) that

$$T(\lambda) = \gamma\, L_{2k+1,2k}(\lambda)\, L_{2k,2k+1'}(\lambda)\, L_{2k+1',2k}(\lambda) \left(\overset{\frown}{\prod_{j=2,\cdots,2k-1}} L_{j+1,j}(\lambda) \right) L_{2,1'}(\lambda)\, L_{1',2}(\lambda)\, L_{2,1}(\lambda), \tag{55}$$

where

$$\gamma = \frac{1}{\ell(\sigma(\alpha_1),\lambda)\ell(\sigma(\alpha_{2k}),\lambda)}.$$ (56)

Using the bulk zero curvature conditions,

$$L_{2j,2j-1}(\lambda)\,L_{2j-1,2j-2}(\lambda) = L_{2j,2j-1'}(\lambda)\,L_{2j-1',2j-2}(\lambda),$$ (57)

for $j = 2,\ldots,k$, and then

$$L_{2j+1',2j}(\lambda)\,L_{2j,2j-1'}(\lambda) = L_{2j+1',2j'}(\lambda)\,L_{2j',2j-1'}(\lambda),$$ (58)

for $j = 1,\ldots,k$, we find

$$T(\lambda) = \gamma\,L_{2k+1,2k}(\lambda)\,L_{2k,2k+1'}(\lambda)\,T'(\lambda)\,L_{1',2}(\lambda)\,L_{2,1}(\lambda).$$ (59)

Similarly for $\check{T}(\eta)$, one has

$$\check{T}(\sigma(\lambda)) = \check{\gamma}\,L_{1,2}(\sigma(\lambda))\,L_{2,1'}(\sigma(\lambda))\,\check{T}'(\sigma(\lambda))\,L_{2k+1',2k}(\sigma(\lambda))\,L_{2k,2k+1}(\sigma(\lambda)),$$ (60)

where

$$\check{\gamma} = \frac{1}{\ell(\sigma(\alpha_1),\sigma(\lambda))\ell(\sigma(\alpha_{2k}),\sigma(\lambda))}.$$ (61)

Therefore,

$$\mathcal{E}\mathcal{T}(\lambda) = \gamma\check{\gamma}\left[L_{1',2}(\lambda)L_{2,1}(\lambda)K_-(x_1;\sigma(\lambda))\,L_{1,2}(\sigma(\lambda))\,L_{2,1'}(\sigma(\lambda))\right]\check{T}'(\sigma(\lambda))$$
$$\times \left[L_{2k+1',2k}(\sigma(\lambda))\,L_{2k,2k+1}(\sigma(\lambda))K_+(x_{2k+1},\lambda)L_{2k+1,2k}(\lambda)L_{2k,2k+1'}(\lambda)\right]T'(\lambda)\,\mathcal{E}.$$ (62)

Taking (50) and (51) into account, the terms in square brackets can be reduced to

$$L_{1',2}(\lambda)L_{2,1}(\lambda)K_-(x_1;\sigma(\lambda))\,L_{1,2}(\sigma(\lambda))\,L_{2,1'}(\sigma(\lambda)) = \ell(\alpha_1,\sigma(\lambda))\ell(\sigma(\alpha_1),\sigma(\lambda))\epsilon_-K_-(x_1';\sigma(\lambda)),$$ (63)

$$L_{2k+1',2k}(\sigma(\lambda))\,L_{2k,2k+1}(\sigma(\lambda))K_+(x_{2k+1},\lambda)L_{2k+1,2k}(\lambda)L_{2k,2k+1'}(\lambda)$$
$$= \ell(\sigma(\alpha_{2k}),\lambda)\ell(\alpha_{2k},\lambda)\epsilon_+K_+(x_{2k+1}',\lambda).$$ (64)

Matching all the factors completes the proof. ∎

Corollary 3.2.   In the case of autonomous maps (39–40), or, in the nonautonomous case, if $\ell(\alpha, \lambda)$ is such that $\ell(\alpha, \sigma(\lambda))/\ell(\sigma(\alpha), \lambda) = 1$, then function $\mathfrak{t}(\lambda) = \mathrm{Tr}\,\mathcal{T}(\lambda)$ satisfies

$$\mathfrak{t}'(\lambda) = \epsilon_-\epsilon_+\, \mathfrak{t}(\lambda) \tag{65}$$

and thus can be taken as a generating function for the invariants of the corresponding map. In the case $\epsilon_-\epsilon_+ = 1$, this is automatic. In the case $\epsilon_-\epsilon_+ = -1$, a quantity I extracted from $\mathfrak{t}(\lambda)$ is a 2-integral. To obtain an invariant, it suffices to take I^2 for instance.

Proof.   In the autonomous case, we have $\alpha_1 = \alpha = \sigma(\alpha_{2k})$; hence,

$$\frac{\ell(\sigma(\alpha_1), \lambda)\ell(\sigma(\alpha_{2k}), \sigma(\lambda))}{\ell(\alpha_1, \sigma(\lambda))\ell(\alpha_{2k}, \lambda)} = 1. \tag{66}$$

The latter is also true in the nonautonomous case if $\ell(\alpha, \sigma(\lambda))/\ell(\sigma(\alpha), \lambda) = 1$. Therefore, in both cases, (53) reduces to

$$\mathcal{T}'(\lambda)\mathcal{E} = \epsilon_-\,\epsilon_+\,\mathcal{E}\,\mathcal{T}(\lambda), \tag{67}$$

and the result follows. ∎

With general lattice parameters along the staircase, the maps (45) and (43) are nonautonomous. However, the $(n-1)$-st power of these maps are autonomous, and we obtain results similar to Corollary 3.2. The double-row monodromy matrix provides k-integrals for the nonautonomous maps, where $k = n - 1$ or $k = 2(n-1)$ when n even and $\epsilon_-\epsilon_+ = -1$ (k-integrals were introduced in [17]).

Corollary 3.3.   Let $\mathcal{T}^{(n-1)}$ denote the monodromy matrix as defined in (49) after the $(n-1)$-st iterate of the map. Then, $\mathcal{T}$ satisfies

$$\mathcal{T}^{(n-1)}\mathcal{E}_n = (\epsilon_-\epsilon_+)^{n-1}\mathcal{E}_n\mathcal{T}, \tag{68}$$

where

$$\mathcal{E}_n = \mathcal{E}^{(n-2)}\mathcal{E}^{(n-3)}\ldots\mathcal{E}^{(1)}\mathcal{E}, \tag{69}$$

and

$$\mathcal{E}^{(j)} = L(x_1^{(j+1)}, x_2^{(j)}; \sigma(\alpha_1^{(j)}), \lambda) L(x_2^{(j)}, x_1^{(j)}; \alpha_1^{(j)}, \lambda), \quad 1 \leq j \leq n-2, \tag{70}$$

with the superscript j denoting a j-step evolution of the associated fields and parameters.

Proof.   It suffices to show that

$$\frac{\ell(\sigma(\alpha_1^{(n-2)}), \lambda)\ell(\sigma(\alpha_{2k}^{(n-2)}), \sigma(\lambda))}{\ell(\alpha_1^{(n-2)}, \sigma(\lambda))\ell(\alpha_{2k}^{(n-2)}, \lambda)} \frac{\ell(\sigma(\alpha_1^{(n-1)}), \lambda)\ell(\sigma(\alpha_{2k}^{(n-1)}), \sigma(\lambda))}{\ell(\alpha_1^{(n-1)}, \sigma(\lambda))\ell(\alpha_{2k}^{(n-1)}, \lambda)}$$

$$\cdots \frac{\ell(\sigma(\alpha_1), \lambda)\ell(\sigma(\alpha_{2k}), \sigma(\lambda))}{\ell(\alpha_1, \sigma(\lambda))\ell(\alpha_{2k}, \lambda)} = 1. \tag{71}$$

Then, (68) follows directly from (53) and its updates. The above equality involves parameters α_1, α_{2k} and their updates. Let us consider the odd case $n = 2k+1$. There are $2k$ parameters along the staircase, namely $\alpha_1, \alpha_2, \ldots, \alpha_{2k}$. One could make the following identification between α_1, α_{2k} and their updates:

$$\alpha_1^{(1)} = \alpha_3, \ \alpha_1^{(2)} = \alpha_5, \ \ldots \ , \alpha_1^{(k-1)} = \alpha_{2k-1}, \ \alpha_1^{(k)} = \sigma(\alpha_{2k}), \ \alpha_1^{(k+1)} = \sigma(\alpha_{2k-2}),$$

$$\ldots \ , \alpha_1^{(2k-1)} = \sigma(\alpha_2),$$

$$\alpha_{2k}^{(1)} = \alpha_{2k-2}, \ \alpha_{2k}^{(2)} = \alpha_{2k-4}, \ \ldots \ , \alpha_{2k}^{(k-1)} = \alpha_2, \ \alpha_{2k}^{(k)} = \sigma(\alpha_1), \ \alpha_{2k}^{(k+1)} = \sigma(\alpha_3),$$

$$\ldots \ , \alpha_{2k}^{(2k-1)} = \sigma(\alpha_{2k-1}).$$

The equality (71) follows from the above identifications. The even case can be proved in a similar way. ∎

In general, the operation of extracting invariants from $\mathfrak{t}(\lambda)$ has to be done carefully as it assumes that there is a natural way to expand $\mathfrak{t}(\lambda)$ in λ. The normalisation of L and K as well as their dependence on λ play a role as they could lead to $\mathfrak{t}(\lambda)$ not being a Laurent polynomial in λ for instance (which is the simplest case for extraction). We will illustrate the procedure on examples for which this can be done relatively easily.

4   Examples of Open Boundary Reductions for the H1 and Q1($\delta = 0$) Equations

Following the construction of open boundary reductions on a strip of the $\mathbb{Z}^2$-lattice, we provide some explicit maps of dimension $n \leq 4$ for the H1 and Q1($\delta = 0$) equations

from the ABS list (Section 2.5 for their explicit forms and their Lax and boundary matrices). The invariants of these maps are obtained by taking the trace of the double-row monodromy matrix defined in (49). Each map we compute here possesses enough invariants, which suggest that they are integrable.

4.1   H1 additive case with two different boundary equations

Consider the case where q_- and q_+ are different. We use

$$q_-(x,y,z;\alpha) = x + z, \quad q_+(x,y,z;\alpha) = y(z-x) + \alpha - \mu, \tag{72}$$

with $\sigma(\alpha) = -\alpha + 2\mu$ (Table 1). First, we consider the maps (39) and (40) with lattice parameters $\alpha, \sigma(\alpha)$. However, a more convenient parameter is $c = 2(\mu - \alpha)$.

The 2D map (40)

$$(x_1, x_2) \mapsto \left(-x_1, x_2 + \frac{c}{2x_1} \right) \tag{73}$$

is an involution.

The 3D map (39) is given by

$$(x_1, x_2, x_3) \mapsto \left(-x_1, x_2 + \frac{2cx_2}{2x_1x_2 + 2x_2x_3 - c}, x_3 - \frac{c}{2x_2} \right). \tag{74}$$

There is an obvious invariant given by x_1^2. Another independent invariant is obtained using our construction. Using

$$L(x,y;\alpha,\lambda) = \frac{1}{\sqrt{\lambda - \mu}} \begin{pmatrix} y & \alpha - \lambda - xy \\ 1 & -x \end{pmatrix}, \quad K_-(x,\lambda) = \begin{pmatrix} 1 & \frac{\mu-\lambda}{x} \\ 0 & 1 \end{pmatrix}, \quad K_+(x,\lambda) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{75}$$

we find

$$\mathfrak{t}(\lambda) = \frac{I}{\lambda - \mu}, \quad I_1 = \frac{(x_1 + x_3)(2x_2(x_1 - x_3) + c)}{x_1}. \tag{76}$$

Recall that $\epsilon_- = -1 = -\epsilon_+$ here; hence, we know that $I_1' = -I_1$, which indeed can be checked directly. This means that I_1 is a 2-integral. In order to obtain an invariant of the map, one can multiply I_1 by x_1, which is also a 2-integral. In particular, we do not need to consider I_1^2 for instance. Therefore, we have the following two invariants for the

map (74)

$$x_1^2, \quad (x_1 + x_3)(2x_2(x_1 - x_3) + c). \tag{77}$$

In terms of variables $y_1 = x_3$, $y_2 = x_3 - \frac{c}{2x_2}$, $y_3 = x_1$, each map in the one-parameter family of maps (74) is equivalent to

$$\gamma : (y_1, y_2, y_3) \mapsto \left(y_2, \frac{y_1 y_2 - y_1 y_3 + 2y_2 y_3}{y_3 + 2y_1 - y_2}, -y_3 \right), \tag{78}$$

and the preserved integral is

$$J(y) = \frac{(y_2 - y_3)(y_1 + y_3)}{y_1 - y_2}. \tag{79}$$

The map (78) is understood geometrically as $\gamma = \mathfrak{s} \circ \iota$ where $\mathfrak{s} : (y_1, y_2, y_3) \to (y_2, y_1, -y_3)$ is an anti-symmetry switch and $\iota : (y_1, y_2, y_3) \to (y_1', y_2, y_3)$ is an anti-horizontal switch, cf. [14], that is,

$$J(\mathfrak{s}(y)) = -J(y), \quad J(\iota(y)) = -J(y). \tag{80}$$

For fixed y_3 each curve $J(y) = j$ intersects horizontal (and vertical) lines once, ι maps y on $J(y) = j$ to the unique point y' on $J(y) = -j$ that has the same y_2, y_3, and ι being an anti-symmetry switch means that the reflection in the line $y_1 = y_2$ of the line $J(y) = j$ with y_3 fixed equals the line $J(y) = -j$ at $-y_3$.

The 4D map equals

$$(x_1, x_2, x_3, x_4) \to \left(-x_1, x_2 + \frac{c(x_2 - x_4)}{(x_2 - x_4)(x_1 + x_3) - c}, x_3 + \frac{c}{x_4 - x_2}, x_4 + \frac{c(x_2 - x_4)}{2x_3(x_4 - x_2) + 2c} \right). \tag{81}$$

With (75), we find

$$\mathfrak{t}(\lambda) = \frac{I_2}{(\lambda - \mu)^2}, \quad I_2 = \frac{(x_1 + x_3)(2x_3(x_4 - x_2) + c)((x_2 - x_4)(x_1 - x_3) + c)}{x_1}. \tag{82}$$

We have an obvious invariant x_1^2 and another one easily constructed from I_2, which satisfies $I_2' = -I_2$ (since $\epsilon_- \epsilon_+ = -1$), by multiplying it by x_1. In terms of the variables $y_1 = x_3$, $y_2 = x_3 - \frac{c}{x_2 - x_4}$, $y_3 = x_1$, the map can be written as $\mathfrak{s} \circ \iota$, where $\mathfrak{s}(y) = (y_2, y_1, -y_3)$

and $\iota(y) = (y_1', y_2, y_3)$ with

$$y_1' = y_2 \left(-1 + \frac{4y_2(y_1 + y_3)}{3y_1 y_2 + y_1 y_3 - y_2^2 + y_2 y_3} \right) \tag{83}$$

are two involutions that leave

$$\frac{(y_2 - y_3)(y_1 + y_3)(y_1 + y_2)}{(y_1 - y_2)^2} \tag{84}$$

invariant.

Let us now consider the nonautonomous 3D map (41) with general parameters, with $q_- = 0$ on the left boundary and $q_+ = 0$ on the right boundary as before. In that case, the map reads

$$(x_1, x_2, x_3; \alpha_1, \alpha_2) \mapsto \left(-x_1, x_2 + \frac{x_2(\alpha_2 - \alpha_1)}{x_2(x_1 + x_3) + \mu - \alpha_2}, x_3 + \frac{\mu - \alpha_2}{x_2}; \sigma(\alpha_2), \sigma(\alpha_1) \right). \tag{85}$$

With (75), we have

$$L(x, y, \alpha, \lambda) L(y, x, \alpha, \lambda) = \ell(\alpha, \lambda) \mathrm{id}, \quad \ell(\alpha, \lambda) = \frac{\alpha - \lambda}{\lambda - \mu}. \tag{86}$$

Hence, here, $\ell(\alpha, \lambda)$ satisfies

$$\frac{\ell(\alpha, \sigma(\lambda))}{\ell(\sigma(\alpha), \lambda)} = 1, \tag{87}$$

thus ensuring that the ratio in (53) is one. As in the autonomous case, we find that $\mathfrak{t}(\lambda)$ provides us with a 2-integral:

$$\mathfrak{t}(\lambda) = \frac{I_3}{2(\lambda - \mu)}, \quad I_3 = \frac{x_2({x_1}^2 - {x_3}^2) + \mu(x_1 - x_3) - \alpha_1 x_1 + \alpha_2 x_3}{x_1}, \tag{88}$$

and a direct calculation gives $I_3' = -I_3$. With the same reasoning as before, we get the following two invariants:

$$x_1^2, \quad x_2({x_1}^2 - {x_3}^2) + \mu(x_1 - x_3) - \alpha_1 x_1 + \alpha_2 x_3.$$

The square of this map leaves x_1, α_1, α_2 invariant, and its action on x_2, x_3 is

$$\begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} \frac{(x_1 x_2 + x_2 x_3 + \mu - \alpha_1)(x_1{}^2 x_2{}^2 - x_2{}^2 x_3{}^2 + \mu x_1 x_2 - \mu x_2 x_3 - \alpha_1 x_1 x_2 + \alpha_1 x_2 x_3 + \mu \alpha_1 - \mu \alpha_2 - \alpha_1 \alpha_2 + \alpha_2{}^2)}{(x_1{}^2 x_2 - x_2 x_3{}^2 + \mu x_1 - \mu x_3 - 2\alpha_1 x_1 + \alpha_2 x_1 + \alpha_2 x_3)(x_1 x_2 + x_2 x_3 + \mu - \alpha_2)} \\ x_3 + \frac{(x_1 + x_3)(\alpha_1 - \alpha_2)}{x_1 x_2 + x_2 x_3 + \mu - \alpha_1} \end{pmatrix}. \tag{89}$$

In terms of variables $y_1 = x_3$, $y_2 = x_3 + \frac{(x_1 + x_3)(\alpha_1 - \alpha_2)}{x_1 x_2 + x_2 x_3 + \mu - \alpha_1}$ this map reads

$$\delta : (y_1, y_2) \mapsto \left(y_2, -\frac{x_1{}^2 y_1 - 2 x_1{}^2 y_2 + y_1 y_2{}^2}{x_1{}^2 - 2 y_1 y_2 + y_2{}^2} \right), \tag{90}$$

which leaves invariant the ratio

$$R = \frac{x_1{}^2 - y_1 y_2}{y_1 - y_2}. \tag{91}$$

The map (90) is similar to the map (78), it can be written as $\delta = \mathfrak{s} \circ \iota$ where $\mathfrak{s} : (y_1, y_2) \to (y_2, y_1)$ is an anti-symmetry switch for R and $\iota : (y_1, y_2) \to (y_1', y_2)$ is the anti-horizontal switch. We note that the degree growth of these maps is linear, which indicates that they are linearisable.

Invariants can be calculated using computer algebra for n-dimensional maps with $n \leq 7$ quite easily. For H1 with two different boundary equations, as in (72), we found $\lfloor (n + 1)/2 \rfloor$ functionally independent integrals.

The occurrence of "anti-switches" seems to be new, at least they do not arise when taking periodic reductions. For comparison, the 3D (2,1)-periodic reduction of H1 is

$$(x_1, x_2, x_3) \to \left(x_2, x_3, x_1 + \frac{\alpha - \beta}{x_3 - x_2} \right).$$

In terms of $u = x_2 - x_1$, $v = x_3 - x_2$, the map reduces to

$$\gamma : (u, v) \to \left(v, -u - v + \frac{\alpha - \beta}{v} \right),$$

which is a root of a QRT mapping, as it can be written as $\gamma = \sigma \circ \iota_1 = \iota_2 \circ \sigma$ in terms of the symmetry switch $\sigma(u, v) = (v, u)$ and the horizontal and vertical switches, ι_1, ι_2, preserving the biquadratic product $(u + v)(uv - \alpha + \beta)$. The 4D (3,1)-periodic reduction of H1 is linearisable; see [20, Section 3.3].

4.2   Q1($\delta = 0$) multiplicative case with two different boundary equations

Consider the Q1($\delta = 0$) equation in the form (37) with two different boundary equations taken from Table 3. We use the following $q_{\pm}$ under the multiplicative involution $\sigma(\alpha) = \frac{\mu^2}{\alpha}$:

$$q_-(x,y,z,\alpha) = \alpha^2(x-y) + \mu^2(y-z), \quad q_+(x,y,z,\alpha) = \alpha^2(x-y)z + \mu^2(y-z)x. \qquad (92)$$

We use the general lattice parameters in this example. The 2D map reads

$$(x_1, x_2) \mapsto \left(\frac{x_1 + (-1 + c^2)x_2}{c^2}, x_2 \frac{(x_1 + (-1 + c^2)x_2)}{c^2 x_1} \right), \qquad (93)$$

where $c = \frac{\mu}{\alpha}$. An N-step iteration of the map yields

$$(x_1, x_2) \mapsto \left(x_1 Y^N, x_2 Y^N \right), \quad Y = \frac{x_1 + (c^2 - 1)x_2}{c^2 x_1}. \qquad (94)$$

By taking the trace of the monodromy matrix, one obtains one invariant x_1/x_2, which can be easily checked by the above general expression of the maps.

The 3D map with generic parameters α_1, α_2 is

$$(x_1, x_2, x_3) \mapsto \left(c_1^2(x_1 - x_2) + x_2, x_2 \frac{x_2^2 + c_1^2(x_1 - x_2)(x_2 - x_3) + x_1 x_3 - 2x_2 x_3}{x_2^2 + c_2^2(x_1 - x_2)(x_2 - x_3) + x_1 x_3 - 2x_2 x_3}, \frac{x_2 x_3}{c_2^2(x_2 - x_3) + x_3} \right), \qquad (95)$$

where $c_j = \frac{\alpha_j}{\mu}, j = 1, 2$ and the change of parameters $(\alpha_1, \alpha_2) \mapsto (\sigma(\alpha_2), \sigma(\alpha_1))$, that is, $(c_1, c_2) \mapsto (1/c_2, 1/c_1)$ is understood. The boundary matrices are given in Table 3,

$$K_-(x, \lambda) = \begin{pmatrix} -\frac{\lambda}{\mu} & \frac{(\lambda^2 + \mu^2)x}{\lambda \mu} \\ 0 & \frac{\mu}{\lambda} \end{pmatrix}, \quad K_+(x, \lambda) = \begin{pmatrix} -\frac{\mu}{\lambda} & 0 \\ -\frac{\lambda^2 + \mu^2}{\lambda \mu x} & \frac{\lambda}{\mu} \end{pmatrix}. \qquad (96)$$

The Lax matrix (38) with scaling (27) yields

$$L(x, y; \alpha, \lambda) = \frac{\mu}{\alpha^2(x - y)\sqrt{\lambda^2 - \mu^2}} \begin{pmatrix} \alpha^2(x - y) - \lambda^2 x & \lambda^2 xy \\ -\lambda^2 & \alpha^2(x - y) + \lambda^2 y \end{pmatrix}, \qquad (97)$$

which satisfies $L(x, y; \alpha, \lambda)L(y, x; \alpha, \lambda) = \ell(\alpha, \lambda)$id with $\ell(\alpha, \lambda) = \frac{\mu^2(\alpha^2 - \lambda^2)}{\alpha^2(\lambda^2 - \mu^2)}$. This implies that

$$\frac{\ell(\alpha, \sigma(\lambda))}{\ell(\sigma(\alpha), \lambda)} = \frac{\mu^2}{\alpha^2}, \tag{98}$$

and hence that the ratio in (53) is nontrivial. In this example, it is possible to further rescale $L \to \zeta(\alpha)L = \mathcal{L}$, by a function of α only, without changing the bulk zero curvature or the boundary zero curvature equations. We introduce

$$\mathcal{L}(x, y; \alpha, \lambda) = \sqrt{\alpha}L(x, y; \alpha, \lambda). \tag{99}$$

This gives $\mathcal{L}(x, y; \alpha, \lambda)\mathcal{L}(y, x; \alpha, \lambda) = \ell^*(\alpha, \lambda)$id where $\ell^*(\alpha, \lambda) = \frac{\mu^2(\alpha^2 - \lambda^2)}{\alpha(\lambda^2 - \mu^2)}$ now satisfies

$$\frac{\ell^*(\alpha, \sigma(\lambda))}{\ell^*(\sigma(\alpha), \lambda)} = 1, \tag{100}$$

as desired. Equipped with this $\mathcal{L}$ and $K_{\pm}$, the trace of the double-row monodromy matrix is invariant (recall that $\epsilon_-\epsilon_+ = 1$ here). We find

$$\mathfrak{t}(\lambda) = \frac{\mu^2}{(\lambda^2 - \mu^2)^2}\left((\lambda^2 + \mu^2)^2 C + 2\lambda^2\mu^2\frac{(c_1^2 + 1)(c_2^2 + 1)}{c_1 c_2}\right), \tag{101}$$

where

$$C = \frac{c_2^2 x_1(x_2 - x_3)^2 + c_1^2(x_1 - x_2)(c_2^2(x_1 - x_3)(x_2 - x_3) + (x_1 - x_2)x_3)}{c_1 c_2(x_1 - x_2)(x_2 - x_3)x_3}. \tag{102}$$

The nonautonomous 3D map can be reduced to a nonautonomous 2D map. Using reduced variables, $z_1 = (x_1 - x_2)/x_3$, $z_2 = x_2/x_3 - 1$, we get

$$(z_1, z_2, c_1, c_2) \mapsto \left(\frac{z_1(1 + c_2^2 z_2)(c_2^2 z_2(1 + z_2) + c_1^2(z_1 + c_2^2 z_1 z_2))}{(1 + z_2)(z_1 + z_2 + c_2^2 z_1 z_2 + z_2^2)},\right.$$
$$\left.\frac{z_2(c_2^2 z_2(1 + z_2) + c_1^2(z_1 + c_2^2 z_1 z_2))}{z_1 + z_2 + c_2^2 z_1 z_2 + z_2^2}, \frac{1}{c_2}, \frac{1}{c_1}\right), \tag{103}$$

and a reduced invariant

$$C = \frac{c_2^2 z_2^2(1 + z_1 + z_2) + c_1^2 z_1(z_1 + c_2^2 z_1 z_2 + c_2^2 z_2^2)}{c_2^2 z_1 z_2}. \tag{104}$$

The square of (103) is an autonomous map of the plane, which is written in terms of $x = z_1, y = z_2, \alpha = c_1^2, \beta = c_2^2$ as

$$\gamma : (x, y) \mapsto \frac{(x + y)(\alpha x + \beta(\alpha x + y + 1)y)^2}{\beta\, (x + (x\beta + y + 1)y)\,((\alpha^2 + \beta)xy^2 + \alpha(\beta x^2 + y^2)y + \alpha(x + y)^2)}$$
$$\left(\frac{x\,(\alpha x + (\alpha\beta x + \beta y + \alpha)y)}{\alpha\,(x + (\alpha x + y + 1)y)}, Y\right). \tag{105}$$

It leaves invariant the pencil of curves of genus 0, cf. expression (104),

$$y^2(1 + x + y) + \alpha x\left(\frac{x}{\beta} + xy + y^2\right) = Cxy, \tag{106}$$

where C is now a parameter distinguishing the curves in the pencil. The map γ can be understood geometrically as the composition of two so-called *p*-switches (see [23] where this terminology was introduced),

$$\gamma = \iota_q \circ \iota_p. \tag{107}$$

A *p*-switch ι_p maps a point r on a curve of the pencil to the 3rd point in the intersection of the curve with the line through r and the involution point p. The pencil (106) has five base points, in homogeneous coordinates:

$$(0 : 0 : 1),\ (0 : -1 : 1),\ (1 : 0 : 0),\ (1 : -1 : 0),\ (1 : -c_1^2 : 0), \tag{108}$$

of which the 1st one is singular (with multiplicity 2). In formula (107), exactly one of p or q should be a non-singular base point of the pencil. The other point lies on an *involution curve*, that is, the involution point depends on the curve in the pencil. For example, if we take $p = (0, -1)$ (the 2nd base point in the above list), then q is given by

$$q = \left(\frac{(C\alpha - C\beta + \alpha^2 - \beta^2)\,\alpha}{(-C\alpha^2 + C\alpha\,\beta - 2\,\alpha^3 + 2\,\alpha^2\beta + C^2 + 4C\alpha + 4\,\alpha^2)\,\beta}, -\frac{\alpha\,(\alpha + \beta + C)}{\beta\,(-\alpha^2 + \alpha\,\beta + C + 2\,\alpha)}\right), \tag{109}$$

which is a parametrisation of the dotted curve in Figure 16.

If we choose p to be the base point at $(\infty, 0)$ (the 3rd base point in the above list), then ι_p is the so-called horizontal shift, usually denoted by ι_1. We have $\gamma = \iota_r \circ \iota_1$, where

$$r = \left(-\frac{C^2 + C\alpha + 3C\beta + 2\,\alpha\,\beta + 2\,\beta^2}{C\alpha^2 - C\alpha\,\beta + 2\,\alpha^2\beta - 2\,\alpha\,\beta^2 + \alpha^2 - 2\,\alpha\,\beta + \beta^2}, -\frac{\alpha + \beta + C}{C\alpha + 2\,\alpha\,\beta + \alpha - \beta}\right). \tag{110}$$

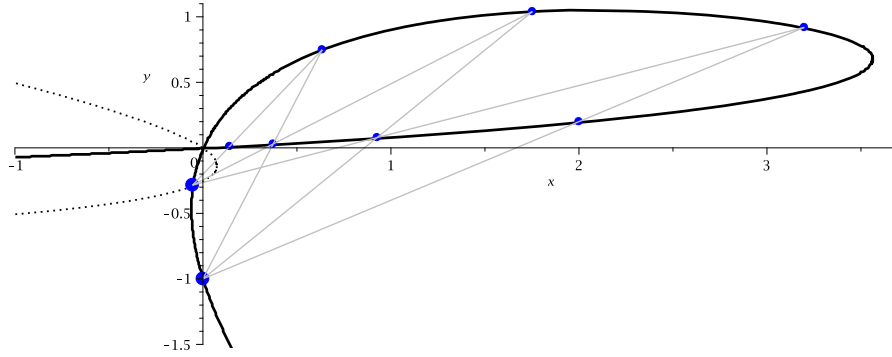More details on these and other involutions with involution curves are provided in [19].

Fig. 16. One of the involution points, $q = (-3850/69789, -154/541)$, lies on an involution curve (dotted), which is given by $(xy\alpha + y^2)\beta + (x+y)\alpha = 0$. We have taken $\alpha = 1/4, \beta = 1$ and $C = 337/100$.

The maps we have obtained in this section are more sophisticated than maps obtained by periodic reduction. The 3D (2,1)-periodic reduction of Q1($\delta = 0$) is

$$(x_1, x_2, x_3) \rightarrow \left(x_2, x_3, \frac{x_2 \left(x_1 - x_3 \right) a - x_3 \left(x_1 - x_2 \right) b}{\left(x_1 - x_3 \right) a - \left(x_1 - x_2 \right) b} \right).$$

Introducing $z = (x_2 - x_1)/(x_3 - x_2)$ this map reduces to a one-dimensional fractional linear map,

$$z \rightarrow \frac{bz}{a(z+1)} - 1.$$

The 4D (3,1)-periodic reduction of Q1($\delta = 0$) is

$$(x_1, x_2, x_3, x_4) \rightarrow \left(x_2, x_3, x_4, \frac{x_2 \left(x_1 - x_4 \right) a - x_4 \left(x_1 - x_2 \right) b}{\left(x_1 - x_4 \right) a - \left(x_1 - x_2 \right) b} \right).$$

In terms of dimension reducing variables $u = (x_2 - x_1)/(x_3 - x_2)$, $v = (x_1 - x_4)/(x_3 - x_4)$, we obtain

$$(u, v) \rightarrow \left(\frac{v - 1}{u - 1}, \frac{bu(v - 1)}{av(u - 1)} \right). \tag{111}$$

The monodromy matrix provides us with a biquadratic invariant, which in terms of u, v equals $B = (av^2 + bu^2)/(uv)$. We note that the curves in the invariant pencil defined by B are reducible (products of lines through 0). Although (111) leaves invariant a biquadratic, the map is not of QRT type(the QRT map that leaves B invariant scales
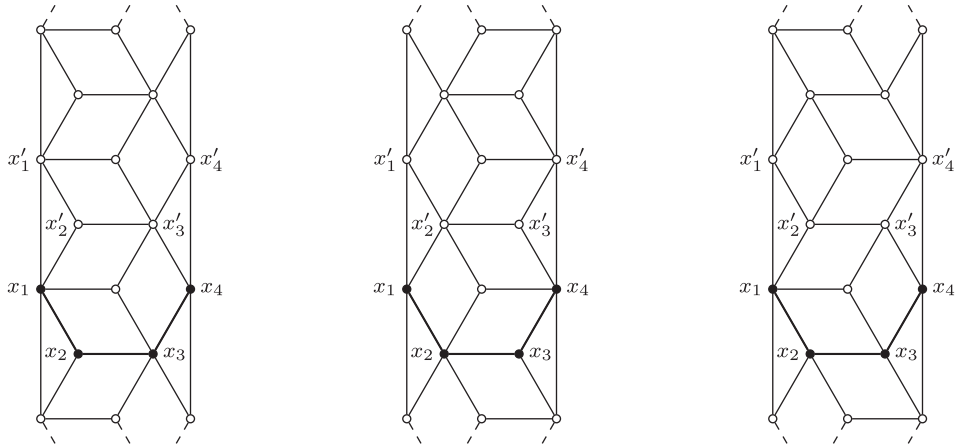
Fig. 17. Open boundary reduction on a hexagonal lattice. The three configurations amount to the same map due to the 3D-consistency of the bulk equation.
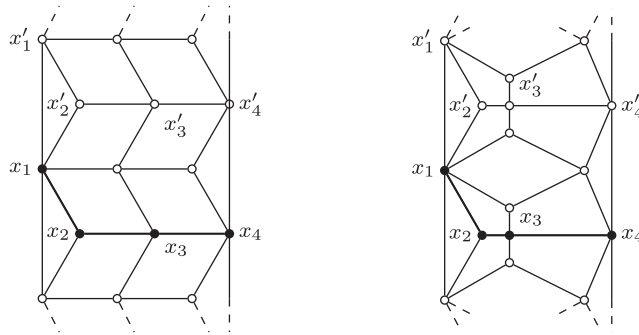

Fig. 18. Other possibilities of open boundary reductions beyond $\mathbb{Z}^2$-lattice.

(u, v) by $av^2/(bu^2)$. Any scaling leaves B invariant.). The map (111) can be written as a composition of a scaling (s denotes the map $(u, v) \rightarrow (su, sv)$) and a p-switch as follows:

$$\iota_{-1,b/a} \circ \frac{b(y-1)}{ay^2 + bx}$$

and

$$\frac{y-1}{x(x+1)} \circ \iota_{-1,1}.$$

In terms of $w = u/v$ the map becomes $w \rightarrow a/(bw)$, which leaves $a/w + bw$ invariant.

5   Concluding Remarks

We introduced the notion of open boundary reductions as a new scheme to construct mappings from integrable initial value problems for quad-graph systems on a strip. This represents an alternative to the well-known periodic reductions. One key ingredient is the idea of dual boundary equations and dual boundary consistency. This allowed us to formulate a boundary zero curvature representation and hence prove that the discrete time evolution of the double-row monodromy matrix is an isospectral deformation (after sufficiently many iterations). The spectral functions obtained by taking the trace of the double-row monodromy matrix are invariants or k-invariants of the discrete n-dimensional mappings, where k is either $2, n-1$, or $2(n-1)$. In contrast with periodic reductions where the underlying graph forms a cylinder, open boundary reductions are built on a strip. The effect of the boundary conditions, for instance via the boundary parameter μ, can be seen explicitly in the maps obtained. Our method leads to different types of maps compared with the periodic reduction method.

There are several natural continuations of the present work. Two pressing questions would be, on the one hand, to investigate discrete Liouville integrability of our maps by introducing an appropriate Poisson structure and, on the other hand, to construct solutions of open boundary discrete problems using the Bäcklund transformations described in [11]. In the continuous realm, the latter point has been successfully carried out by using a nonlinear version of the mirror image method [5, 16]. The nonlinear superposition principle is implemented with Bäcklund transformations and is then used in conjunction with the inverse scattering method to construct solutions. Whether it is possible to obtain such a mapping to a full line problem for a problem on the interval, as a solution method, remains an open problem. In the present discrete context, the appearance of the double-row monodromy matrix suggests that a similar construction should be possible in principle by mapping the problem on a strip (or at least on the half lattice) to a problem on the full lattice. This is a completely open topic. The problem of classifying integrable boundary equations for the ABS equations is still pending, but the duality property seems to be a promising avenue. More generally, classifying, or at least finding, integrable boundary consistency for quad equations beyond the ABS-list (e.g., Boussinesq-type quad equations) is a tantalising prospect. It would also be desirable to investigate more thoroughly the maps we obtained, as well as producing more examples, and to establish a more precise connection with some known examples, for instance, the discrete Painlevé type equations. Finally, equipped with bulk and boundary equations, initial value problems on quad-graphs with boundary can be

naturally formulated, similar to how this is done for quad equations without boundary [3, 22], with characteristic lines reflecting off the boundary. Some examples of well-posed initial boundary-value problems beyond the $\mathbb{Z}^2$-lattice are shown in Figures 17 and 18. A general criterion of the well-posedness of initial-boundary data on generic quad-graphs with boundary remains to be investigated.

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